

Elastic Curves via Geometric Mechanics

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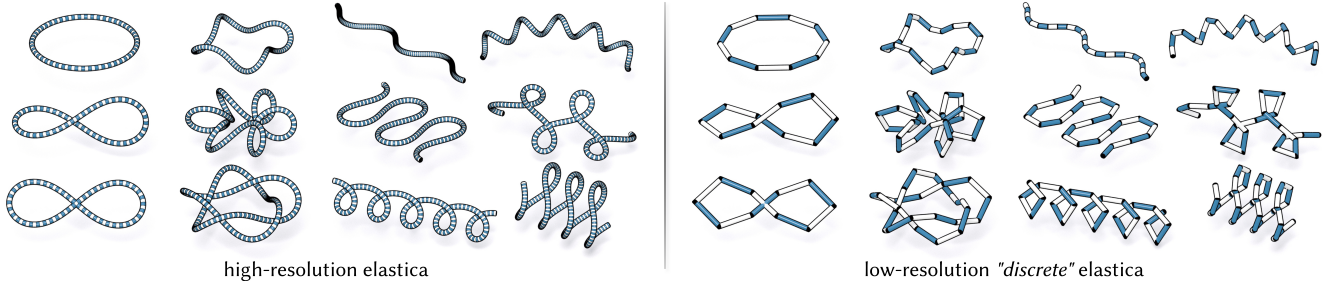


Fig. 1. We compute elastic curves from an *isoperimetric characterization* of smooth elastica that carries over to polygonal curves without requiring auxiliary discretizations of curvature or material frames. The resulting lower-order variational problem can be optimized directly over polygon vertices, is better conditioned, and produces consistent curves across resolutions, providing a canonical notion of *discrete elastica* that retains connections to dynamical systems of the smooth theory.

Elastic curves are the mathematical shapes of thin elastic rods in equilibrium, with deep connections to mechanics, geometry, and computer graphics. Traditionally described as stationary points of bending energy under length and torsion constraints, their rich theory admits many equivalent characterizations. We develop a new one from the viewpoint of geometric mechanics. Our main contribution relies on a lesser-known *isoperimetric characterization*: a curve is elastic if and only if it is a critical point of the length functional under fixed area and volume vectors. We show that these constraints transform naturally under orientation-preserving rigid body motions, identifying them as momentum variables for these symmetries. This structure suggests a new discrete theory. We show that the low-order integral quantities length, area, and volume vectors are all naturally defined for polygonal curves, leaving the same transformation laws exactly satisfied. The resulting definition of discrete elastic curves in terms of the isoperimetric characterization restricted to discrete polygonal curves is variational, structure-preserving, and requires no auxiliary discretizations of curvature or material frames. Finally, the same structure carries the *Marsden–Weinstein* form, a canonical (pre-)symplectic structure on the space of curves, to polygonal curves. This yields novel approaches to Hamiltonian dynamics on discrete space curves, including tangent, vortex-filament, and modified Korteweg–de Vries flows.

Additional Key Words and Phrases: elastic curves, discrete elastica, discrete differential geometry, geometric mechanics, momentum maps, structure-preserving discretization, Hamiltonian curve flows

1 Introduction

Elastic curves, also known as *elastica*, are the shapes taken by thin elastic wires in static equilibrium. The study of elastic curves dates back to the early days of the calculus of variations, beginning with the classical work of Bernoulli and Euler [Euler 1744; Levien 2008]. They continue to play central roles in modeling thin elastic materials, such as in plant growth, DNA mechanics, hair simulation, elastic and magnetic knots, deployable structures and fabrication-aware design [Goriely and Tabor 1998; Benham and Mielke 2005; Bertails et al. 2006; Ricca and Maggioni 2018; Hafner and Bickel 2021, 2023; Panetta et al. 2019; Vidulis et al. 2023; Dandy et al. 2024].

While they are traditionally characterized as stationary points of the bending energy

$$\frac{1}{2} \int \kappa^2 ds,$$

where κ is the curvature, and ds is the arc-length measure of the curve, possibly under constraints such as fixed length and total torsion, there are unusually many equivalent characterizations of the same curves. Kirchhoff’s analogy interprets elastic space curves as spinning-top dynamics of their tangent indicatrices [Kirchhoff 1859], which can be expressed as a pendulum equation [Pinkall and Gross 2024]. They also arise as solitons of the localized induction equation, an integrable curve flow related by the Hasimoto transform to the nonlinear Schrödinger equation [Hasimoto 1971, 1972; Langer and Singer 1984, 1996; Singer 2008; Pinkall and Gross 2024]. Elastic curves also provide a classical bridge to surface theory, where cylinders over free elastica and their Möbius images give Willmore surfaces, with related constructions yielding constrained Willmore surfaces [Pinkall 1985; Bohle et al. 2008; Heller 2014; Pinkall and Gross 2024]. Notably, a number of these results generalize to elastic curves in arbitrary dimensions [Tjaden 1991; Pinkall and Gross 2024], space forms with curvature [Langer and Singer 1984], or non-constant bending stiffness, as in rods of variable thickness [Hafner and Bickel 2021, 2023; Gross et al. 2025].

While these perspectives coincide in the continuous theory, this coincidence is no longer automatic when curves are replaced by polygons, which have well-defined edge lengths and turning angles, but no canonical curvature. From the viewpoint of *Discrete Differential Geometry (DDG)*, one defines discrete quantities and identities that remain meaningful and can be evaluated on polygonal curves [Bobenko et al. 2008; Wardetzky et al. 2007]. For elastic polygons, existing discretizations do so by choosing one of the equivalent continuous characterizations as the basis for the definition.

This choice leads to two broad DDG approaches to elastica. The first takes the local variational characterization as its starting point, defining discrete curvature, frame, and strain quantities directly on polygons. A particularly influential model was proposed by Bergou

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et al. [2008]: discrete elastic rods discretize the local framed-rod picture through adapted frames, parallel transport, holonomy, bending, and torsion, thereby defining a discrete elastic energy for bendable rods. Related framed-rod and Cosserat models extend this viewpoint to stretch, shear, dynamics, contact, and strand simulation [Pai 2002; Kugelstadt and Schömer 2016; Gazzola et al. 2018; Bertails et al. 2006; Derouet-Jourdan et al. 2010; Casati and Bertails-Descoubes 2013; Derouet-Jourdan et al. 2013; Shi et al. 2023; Daviet 2023; Hsu et al. 2025], with applications in hair simulation, inverse design, and fabrication [Panetta et al. 2019; Pillwein et al. 2020; Pillwein and Musialski 2021; Ren et al. 2022; Becker et al. 2023; Vidulis et al. 2023; Dandy et al. 2024; Hafner and Bickel 2021, 2023].

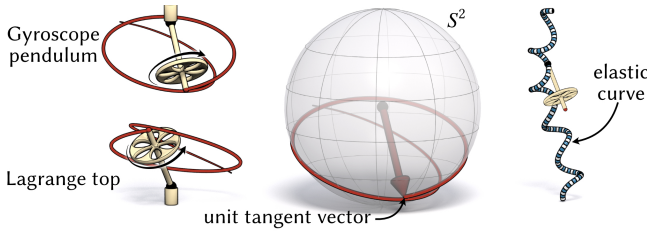


Fig. 2. *Kirchhoff's analogy* identifies the tangent indicatrix of an elastic space curve with the motion of a Lagrange top (equivalently, a symmetric gyroscope under gravity), whose reduced dynamics are governed by a pendulum equation.

A different line preserves the integrable-system characterizations instead. The discrete integrable geometry of curves and lattices was developed in this direction by Doliwa and Santini [1999] and provides semi-discrete analogues of curve flows related to the nonlinear Schrödinger and localized-induction hierarchies. Building on this viewpoint, Bobenko and Suris [2000] introduced an integrable variational discretization of elastic curves through a discrete Lagrange-top formulation (Fig. 2). Pinkall et al. [2007] later constructed a doubly discrete analogue of the smoke-ring flow. Hoffmann and collaborators developed related constructions through discrete Hasimoto flows, Bäcklund transformations, and skew parallelogram nets, characterizing discrete elastic and constrained elastic curves as invariant curves of integrable discrete curve evolutions [Hoffmann 2008; Hoffmann et al. 2025, 2026]. These constructions preserve the integrable-system character of elastica, but define discrete elastica only implicitly through the particular equations or flows they discretize. The resulting notion is therefore inherited from a chosen model of the smooth dynamics rather than posed intrinsically on polygon space, and it does not yield a geometric problem that can be evaluated and optimized directly on polygon vertices.

In this paper, we propose a new formulation for discrete elastic curves that takes a third route. Rather than improving a local curvature, frame, or strain model, and rather than discretizing a particular integrable system, we start from a lesser-known and only recently discovered variational characterization of smooth elastica [Chern et al. 2020]:

Elastic curves are critical points of length under constraints on their area and volume vectors.

Here, components of area and volume vectors are respectively the projected areas along coordinate axes and the swept volumes rotating about coordinate axes.

Smooth elastica therefore admit an isoperimetric characterization that lowers the order of the variational problem. Whereas bending energy depends on second derivatives and leads to a fourth-order Euler–Lagrange equation, the isoperimetric formulation uses only the first-order quantities length, area, and volume and yields second-order stationary conditions. Unlike bending energy, these quantities restrict canonically to polygons without choosing discrete curvature, frames, or strains. This gives a new notion of discrete elastica with better conditioning and consistent behavior under refinement that retains the geometric structure of the smooth theory.

We show that the area and volume vectors have a kinematic meaning in geometric mechanics: for closed curves, they are the *linear* and *angular momenta* associated with translational and rotational symmetries [Marsden and Weinstein 1983; Shashikanth and Marsden 2003]. Building on this viewpoint, we place the isoperimetric characterization of elastica on a systematic footing, and generalize it to quasi-periodic curves, whose geometry repeats up to a prescribed rigid motion, called the *monodromy*. For each monodromy, we identify the compatible area-volume data as the *Noether charges* associated with the residual rigid-motion symmetries. The same first-principles derivation carries over to polygonal curves and yields their discrete Noether charges. Building on this general discrete isoperimetric theory, we develop a numerical method for computing discrete elastica.

As a further consequence, our geometric-mechanics formulation provides a *pre-symplectic structure* for discrete polygonal curves. In the continuous setting, this structure is given by a closed 2-form known as the *Marsden–Weinstein form*. Its kernel consists of reparametrizations, and it therefore descends to a genuine symplectic form after quotienting by reparametrizations. The Marsden–Weinstein form underlies various Hamiltonian flows, including the aforementioned integrable dynamical system of *vortex-filament* motion and the higher-order *modified Korteweg–de Vries (mKdV)* flows in the same hierarchy, while reparametrizations appear as its degenerate directions.

With the new discrete pre-symplectic structure arising from our discrete theory, we obtain structure-driven discrete analogues of tangent, vortex-filament, and mKdV flows that retain the relevant conservation laws up to a small residual, as demonstrated through comparisons with local and integrable discretizations.

A key benefit of this route is that it does not force us to choose one classical characterization of elastica at the expense of the others. The isoperimetric formulation gives the variational definition, the momentum interpretation keeps the rigid-body-mechanics meaning, and the Marsden–Weinstein form retains the connection to Hamiltonian curve flows and the localized-induction hierarchy.

Remaining closely tied to classical results, this new discrete theory opens up numerous opportunities by enabling unexplored methods for analysis and simulation of related concepts. Specifically, we make the following contributions:

- We give a systematic geometric-mechanics treatment of quasi-periodic elastica with monodromy $g \in \text{SE}(3)$, identify

the compatible rigid-motion symmetries, and show that the corresponding area-volume components are their Noether momenta.

- We provide a structure-preserving discretization for quasi-periodic polygonal curves: discrete area and volume, together with the Marsden–Weinstein form restricted to polygonal curves, satisfy the same transformation laws exactly.
- We define discrete elastic curves as critical points of length at fixed momentum on the space of discrete polygonal curves, yielding a lower-order, curvature-free variational characterization.
- We use the discrete Marsden–Weinstein form to define Hamiltonian flows on polygonal curves, yielding structure-driven analogues of tangent, vortex-filament, and mKdV flows.

Outline. The remainder of the paper is organized as follows: Section 2 reviews the classical variational theory of elastic curves together with equivalent characterizations, including the isoperimetric formulation that motivates our approach. In Section 3, we develop the geometric-mechanics framework and identify the area and volume data as the momentum map comprising the Noether charges associated with rigid-motion symmetries. In Section 4, we transfer these structures to polygonal curves and introduce our main discrete formulation, defining discrete elastica as critical points of length at fixed discrete momentum. In Section 5, we present a simple numerical method to compute discrete elastica with prescribed monodromy and momenta. In Section 6, we use the discrete Marsden–Weinstein structure to construct tangent, vortex-filament, and modified Korteweg–de Vries flows. Finally, in Section 7, we evaluate the discrete formulation numerically and validate its convergence, conservation properties, and agreement with the continuous theory.

2 Elastic Curves

Let $\mathcal{M}$ denote the space of smooth curves $\gamma: I \rightarrow \mathbb{R}^3$, where I is either a compact interval $[0, 1]$, the circle S^1 , or the real line $\mathbb{R}$. A smooth *infinitesimal deformation*, or *variation* (virtual velocity), of γ is a vector field $\dot{\gamma}$ along γ . The *tangent space*¹

$$T_\gamma \mathcal{M} = \{\dot{\gamma}: I \rightarrow \mathbb{R}^3\}$$

consists of all such fields, *i.e.*, smooth maps $\dot{\gamma}: I \rightarrow \mathbb{R}^3$. A curve $\gamma \in \mathcal{M}$ is *arc-length parametrized* if $|\dot{\gamma}| = 1$, so that the parameter $t \in I$ measures distance along γ directly. We work throughout with arc-length parametrized curves and write $(\cdot)' = \frac{d}{ds}$ for the arc-length derivative.

2.1 Geometric functionals on $\mathcal{M}$

To set up the variational definition of elastic curves, we first recall a few basic geometric functionals on $\mathcal{M}$. These are quantities that assign a number to each curve, independently of its parametrization and of the chosen Euclidean frame of $\mathbb{R}^3$. Such functionals often

¹We distinguish between γ as an abstract element of the function space $\mathcal{M}$ and $\tilde{\gamma}: I \rightarrow \mathbb{R}^3$ as the corresponding vector-valued function on I . Thus derived quantities, such as $\tilde{\gamma}'$, $\tilde{\gamma}''$, and higher derivatives, and variations, such as $\dot{\tilde{\gamma}}$ and $\ddot{\tilde{\gamma}}$, are also functions on I . For example, $\dot{\tilde{\gamma}}: I \rightarrow \mathbb{R}^3$ is evaluated at $t \in I$ as $\dot{\tilde{\gamma}}(t) \in \mathbb{R}^3$.

appear as terms in elastic potential energy models for materials whose centerline is described by a curve.

2.1.1 Length and Bending Energy. The simplest geometric quantity associated to a curve γ is its *length*

$$\mathcal{L}(\gamma) = \int_I ds. \quad (1)$$

The next is curvature: the unit tangent $T = \tilde{\gamma}'$ traces a path (the *tangent indicatrix*) on the unit sphere $S^2 \subset \mathbb{R}^3$, and its rate of change $\tilde{\gamma}'' = T'$ measures how sharply the curve bends. The *bending energy*

$$\mathcal{B}(\gamma) = \frac{1}{2} \int_I |\tilde{\gamma}''|^2 ds \quad (2)$$

integrates its squared norm over the curve and models the elastic energy stored in a bent inextensible rod [Antman 2005; Audoly and Pomeau 2010].

We are interested in stationary points of Eq. (2). Without further constraints these are free elastica. If length is imposed as a constraint, the Euler–Lagrange equation acquires a Lagrange multiplier term corresponding to tension [Pinkall and Gross 2024].

2.1.2 Holonomy and total torsion. Both length and bending energy are *geometric invariants* of the curve, *i.e.*, they are unchanged under reparametrization and under changes of frame of $\mathbb{R}^3$. For space curves, there is another geometric invariant measuring “twisting.” As one moves along the curve, normal directions can be parallel transported, and comparing the initial and final normal directions gives a net rotation angle. This angle, the *holonomy of the normal bundle*, equivalently *total torsion* modulo 2π , provides the torsional constraint in the classical theory of space elastica [Bishop 1975].

On each normal plane, the map $T \times \cdot$ acts as rotation by 90° . Thus it plays the role of multiplication by i , and defines a *complex structure* on the normal bundle. This complex structure is preserved by parallel transport [Chern et al. 2020, Sec. 3.2].

Given a pair of reference unit normal vectors $W = (W_0, W_1)$ at $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ the *holonomy* is the unique angle $\Phi_W \in \mathbb{R}/2\pi\mathbb{Z}$ such that

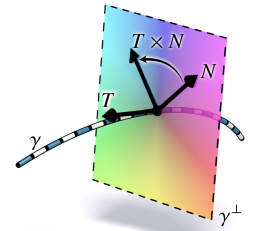
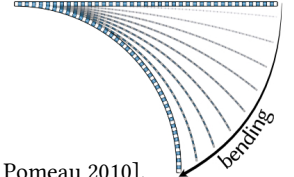
$$Z_1 = e^{-i\Phi_W} W_1 = \cos(\Phi_W) W_1 - \sin(\Phi_W) T_1 \times W_1, \quad (3)$$

where Z_1 is the parallel transport of W_0 along γ . In the closed case $I = S^1$ one takes $W_0 = W_1$. For open curves, Φ_W depends on the choice of endpoint normals W_0 and W_1 . Changing these normals changes only the reference angle between the initial and final normal planes. The variational gradient of Φ_W is independent of this choice [Pinkall and Gross 2024, Thm. 5.3].

Remark 2.1. Much of the literature uses total torsion

$$\mathcal{T}(\gamma, N) = \int_I \langle N', T \times N \rangle ds$$

of a framed curve (γ, N) in place of holonomy. While their variational gradients coincide [Pinkall and Gross 2024], the total torsion is intrinsic to the curve only modulo 2π , which is precisely what the holonomy $\Phi_W \in \mathbb{R}/2\pi\mathbb{Z}$ captures.



2.2 Characterizations of elastic curves

We are now in a position to state the central definition of this section. A curve $\gamma \in \mathcal{M}$ is called *elastic* if it is a critical point of the bending energy $\mathcal{B}(\gamma)$ (Eq. (2)) under the constraints of fixed length $\mathcal{L}_0$ (Eq. (1)) and holonomy Φ_0 (Eq. (3)). The L^2 -gradients of the bending energy, length, and holonomy of an arc-length parametrized curve $\gamma \in \mathcal{M}$ under perturbations with compact support in the interior of I are

$$\begin{aligned}\nabla \mathcal{B} &= \tilde{\gamma}'''' + \frac{3}{2} |\tilde{\gamma}''|^2 \tilde{\gamma}' - \langle \tilde{\gamma}''', \tilde{\gamma}' \rangle \tilde{\gamma}', \\ \nabla \mathcal{L} &= -\tilde{\gamma}'', \\ \nabla \Phi_W &= -\tilde{\gamma}' \times \tilde{\gamma}'''.\end{aligned}$$

Consequently, stationarity of $\mathcal{B} + \lambda_1 \mathcal{L} + \lambda_2 \Phi_W$ yields the Euler-Lagrange equation

$$\tilde{\gamma}'''' + \frac{3}{2} \langle \tilde{\gamma}'', \tilde{\gamma}'' \rangle \tilde{\gamma}' - \langle \tilde{\gamma}''', \tilde{\gamma}' \rangle \tilde{\gamma}' - \lambda_1 \tilde{\gamma}'' - \lambda_2 \tilde{\gamma}' \times \tilde{\gamma}''' = 0, \quad (4)$$

for Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$.

2.2.1 Equivalent characterizations and physical interpretations. Elastic curves admit several equivalent characterizations [Pinkall and Gross 2024] connecting them to mathematical physics and integrable systems, and, as we show in Section 3.2, to symplectic geometry.

Classical examples include *Kirchhoff's analogy* (Fig. 2), which identifies elastic space curves with the motion of a spinning top [Kirchhoff 1859; Bobenko and Suris 2000]. In this description, the tangent indicatrix carries the reduced dynamics and satisfies a pendulum equation.

Another classical characterization is closely tied to vortex-filament dynamics. The elastic Euler-Lagrange Eq. (4) can be written as

$$\frac{d}{ds} (\tilde{\gamma}'''' + \frac{3}{2} |\tilde{\gamma}''|^2 \tilde{\gamma}' - \lambda_1 \tilde{\gamma}' - \lambda_2 \tilde{\gamma}' \times \tilde{\gamma}''') = 0.$$

Hence there is a constant vector $\tilde{c}_1 \in \mathbb{R}^3$ such that

$$\tilde{\gamma}'''' + \frac{3}{2} |\tilde{\gamma}''|^2 \tilde{\gamma}' - \lambda_1 \tilde{\gamma}' - \lambda_2 \tilde{\gamma}' \times \tilde{\gamma}''' = -\tilde{c}_1.$$

Crossing this equation with $\tilde{\gamma}'$ from the left gives

$$\frac{d}{ds} (\tilde{\gamma}' \times \tilde{\gamma}'' + \lambda_2 \tilde{\gamma}') = \tilde{c}_1 \times \tilde{\gamma}'.$$

Integrating once more shows that, for some constant vector $\tilde{c}_2 \in \mathbb{R}^3$ and some $\lambda_3 \in \mathbb{R}$,

$$\tilde{\gamma}' \times \tilde{\gamma}'' = \tilde{c}_1 \times \tilde{\gamma} + \tilde{c}_2 + \lambda_3 \tilde{\gamma}'. \quad (5)$$

Conversely, differentiating this relation and reversing the two steps recovers the elastic Euler-Lagrange equation, so that the two characterizations Eq. (4) and Eq. (5) are indeed equivalent. The left-hand side is the velocity field of the *localized induction equation*

$$\frac{\partial}{\partial t} \tilde{\gamma} = \tilde{\gamma}' \times \tilde{\gamma}'', \quad (6)$$

which governs the evolution of thin vortex filaments and is also known as the *vortex-filament flow*². This flow belongs to a family of mutually commuting curve evolutions—the *localized-induction hierarchy*—meaning that applying two flows from the family in either order gives the same result. Under the *Hasimoto transform*, this family corresponds to the *nonlinear Schrödinger (NLS) hierarchy*. This integrable structure supplies many related curve flows with shared conservation laws [Chern et al. 2020].

²This flow is also known as smoke-ring flow, Hasimoto flow, Heisenberg flow, or binormal curvature flow.

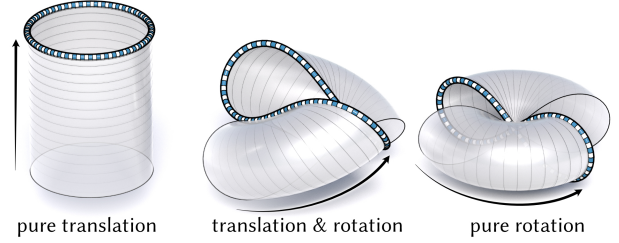


Fig. 3. Under the localized-induction equation, an elastic curve evolves by an orientation-preserving rigid body motion, up to tangential reparametrization. This characterizes elastica as invariant curves of the flow.

Thus, Eq. (5) says that elastic curves are precisely those curves for which the localized induction flow reduces to an infinitesimal rigid motion (Fig. 3), up to the tangential reparametrization term. This viewpoint goes back to Da Rios [1906] and Levi-Civita [1908], was rediscovered repeatedly [Ricca 1991], and was later related by Hasimoto [1971] to the nonlinear Schrödinger equation. It has also motivated discrete theories of elastic curves based on discrete analogues of the vortex-filament flow [Pinkall et al. 2007; Hoffmann 2008]. In these theories, discrete elastica are polygons that evolve by rigid body motions under the chosen discrete flow. Since the continuous velocity is the curvature-binormal vector $\tilde{\gamma}' \times \tilde{\gamma}''$, this approach requires a choice of how curvature, normal directions, and binormal motion are represented on a polygon. Our isoperimetric formulation avoids this choice by using only length, area, and volume. The doubly discrete Hasimoto flow, for example, provides a discrete analogue of the smoke-ring flow, although its variational interpretation is not yet understood.

In contrast to these classical correspondences with spinning tops, pendulums, vortex filaments, and integrable flows, the present work builds on a more recent variational characterization of elastica. First noted by Chern et al. [2020] as a consequence of the recursion formula for the localized-induction hierarchy, this characterization identifies elastica as critical points of the length functional under area and volume constraints. It is well suited to our purposes, since length, area and volume are all canonically defined directly for polygonal curves. In contrast to previous approaches, this “isoperimetric characterization” (see Section 2.2.2) does not require choosing one of the many discrete analogues of curvature, torsion, or integrable system for polygons.

2.2.2 Isoperimetric characterization. The classical *isoperimetric problem* asks which closed planar curve encloses the largest area at fixed length. Equivalently, one may ask which curve has the shortest length at fixed enclosed area. In both forms, the answer is the circle. We now pose an analogous problem for space curves.

For closed space curves $\gamma: S^1 \rightarrow \mathbb{R}^3$, define the *area vector* (Fig. 9)

$$\mathcal{A}(\gamma) = \frac{1}{2} \oint \tilde{\gamma} \times \tilde{\gamma}' ds \in \mathbb{R}^3$$

and the *volume vector* (Fig. 9)

$$\mathcal{V}(\gamma) = \frac{1}{3} \oint \tilde{\gamma} \times (\tilde{\gamma} \times \tilde{\gamma}') ds \in \mathbb{R}^3.$$

The component $\langle \vec{a}, \mathcal{A}(\gamma) \rangle$ of the area vector along a unit vector $\vec{a} \in S^2 \subset \mathbb{R}^3$ equals the signed area of the projection of γ onto $\vec{a}^\perp$. Analogously, the component $\langle \vec{a}, \mathcal{V}(\gamma) \rangle$ of the volume vector equals (up to a factor of 2π) the signed volume swept by rotating a spanning surface around the axis $\mathbb{R}\vec{a}$ (see Section 2.3).

Chern et al. [2020, Cor. 2] show that a curve $\gamma \in \mathcal{M}$ is elastic if and only if it is a critical point of $\mathcal{L}$ at fixed $\mathcal{A}_0$ and $\mathcal{V}_0$. In other words, elastic curves are precisely critical points of the length functional for prescribed area and volume vectors. While at first glance it is not obvious why the area and volume vectors should be the right constraints for a variational problem on space curves, the present paper provides an answer based on the following observation: Under a rigid motion $h = (Q, \vec{c}) \in \text{SE}(3)$, they transform as

$$\begin{aligned} \mathcal{A}(h \circ \gamma) &= Q\mathcal{A}(\gamma), \\ \mathcal{V}(h \circ \gamma) &= Q\mathcal{V}(\gamma) + \vec{c} \times Q\mathcal{A}(\gamma), \end{aligned}$$

which is precisely the coadjoint action of $\text{SE}(3)$ (Section 3.1.2 Eq. (12)). That is, $\mathcal{A}$ and $\mathcal{V}$ respectively transform like linear momentum and angular momentum in 3D kinematics. They constraints are therefore the Noether charges of the rigid-motion symmetry, *i.e.*, the conserved quantities the geometry itself singles out. We develop this connection in more generality in the following Sections 2.3 and 3.

2.3 General Theory

The isoperimetric characterization and the transformation laws for the area and volume vectors $\mathcal{A}$ and $\mathcal{V}$ extend beyond closed curves. To include open curves, we regard a curve segment as one fundamental domain of a *quasi-periodic curve*, by which we mean a curve that repeats up to an orientation-preserving rigid motion $g \in \text{SE}(3)$.

More precisely, let $\mathbb{R}$ have its fixed orientation, and let $\tau: \mathbb{R} \rightarrow \mathbb{R}$ denote a nontrivial translation in the positive direction. We denote by $\mathcal{M}_g$ the space of smooth immersions $\gamma: \mathbb{R} \rightarrow \mathbb{R}^3$ satisfying

$$\tau^* \gamma = g \circ \gamma,$$

or equivalently, $\gamma(\tau(t)) = g(\gamma(t))$. Here, g is the *monodromy*, and any interval $I = [t_0, \tau(t_0)]$ for $t_0 \in \mathbb{R}$ is a *fundamental domain*. The case $g = \text{id}$ recovers closed curves, while general g describes open curves together with a prescribed rigid motion relating consecutive copies.

In this framework, the area and volume vectors admit canonical extensions, and their well-defined components depend only on the

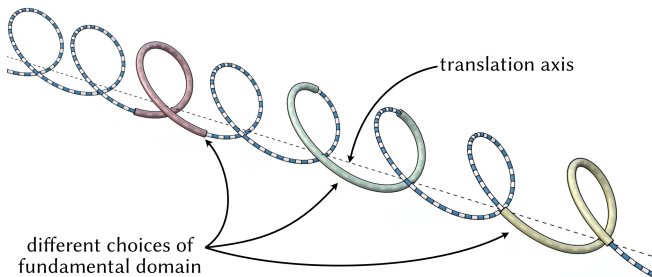


Fig. 4. A quasi-periodic curve with translational monodromy is represented by one fundamental domain, whose copies differ by a fixed translation.

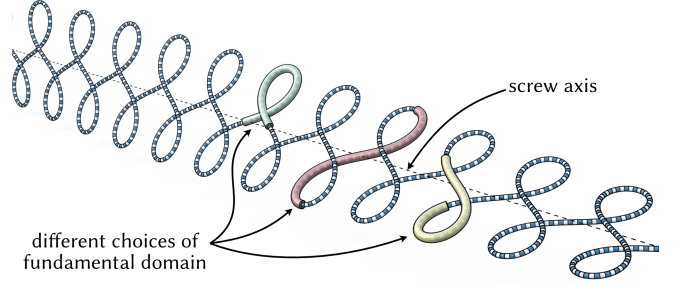


Fig. 5. A quasi-periodic curve with screw monodromy repeats by a rotation and translation along the same axis, so that one fundamental domain represents the full infinite curve.

monodromy. Thus closed and open elastica can be treated within a single framework.

Remark 2.2. By a suitable choice of coordinate origin, the monodromy g always takes the form of a screw motion (see Appendix A), that is, a rotation about and translation along an axis $\mathbb{R}\vec{s}_g$ for some unit vector $\vec{s}_g \in S^2 \subset \mathbb{R}^3$, which we call the screw axis of g .

2.3.1 Monodromy cases and compatible rigid motions. For elements $g \in \text{SE}(3)$ write $g = (R, \vec{b})$, where $R \in \text{SO}(3)$ is a rotation and $\vec{b} \in \mathbb{R}^3$, and distinguish three cases. If $g = \text{id}$, the curve is closed (see Fig. 7). If $g = (\text{Id}, \vec{b})$ with $\vec{b} \neq 0$, consecutive copies differ by a pure translation (see Fig. 4). If $g = (R, \vec{b})$ with $R \neq \text{Id}$, then, after choosing the origin on its axis, g is a screw motion: a rotation about an axis together with a translation along the same axis (see Fig. 5). This distinction determines which components of the area and volume vectors are independent of the chosen fundamental domain.

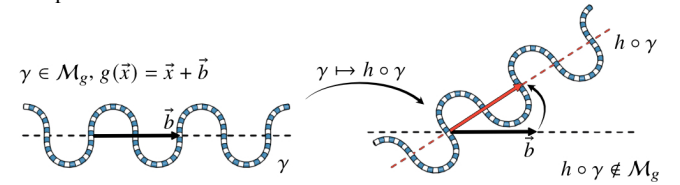


Fig. 6. A rigid motion h generally changes the monodromy of a quasi-periodic curve from g to hgh^{-1} . In the translational example shown, h rotates the translation direction $\vec{b}$, so $h \circ \gamma \notin \mathcal{M}_g$. Thus only rigid motions in $Z(g)$ preserve $\mathcal{M}_g$.

A rigid motion need not preserve the space $\mathcal{M}_g$ (Fig. 6). Indeed, if $h \in \text{SE}(3)$ and $\gamma \in \mathcal{M}_g$, then

$$(h \circ \gamma)(\tau(t)) = h(g(\gamma(t))) = (hgh^{-1})((h \circ \gamma)(t)).$$

Thus $h \circ \gamma \in \mathcal{M}_{hgh^{-1}}$. The rigid motions preserving $\mathcal{M}_g$ are therefore precisely those commuting with g . The corresponding subgroup of $\text{SE}(3)$ is called the *centralizer* of $g \in \text{SE}(3)$ and denoted by

$$Z(g) := \{h \in \text{SE}(3) \mid hg = gh\}.$$

We distinguish three cases for $g = (R, \vec{b}) \in \text{SE}(3)$:

- (i) If $g = \text{id}$, then $Z(g) = \text{SE}(3)$.
- (ii) If $R = \text{Id}$ and $\vec{b} \neq 0$, then

$$Z(g) = \{(Q, \vec{c}) \in \text{SE}(3) \mid Q\vec{b} = \vec{b}, \vec{c} \in \mathbb{R}^3\}.$$

so $Z(g)$ consists of rotations about the direction $\vec{s}_g = \vec{b}/|\vec{b}|$ together with arbitrary translations.

- (iii) If $R \neq \text{Id}$ and g is not a pure half-turn³, choose the origin on its screw axis $\mathbb{R}\vec{s}_g$ and write $\vec{b} = \ell\vec{s}_g$. Then

$$Z(g) = \{(Q, \vec{c}) \in \text{SE}(3) \mid Q\vec{s}_g = \vec{s}_g, \vec{c} \parallel \vec{s}_g\}.$$

2.3.2 Area vector of a space curve. How does one define an area-type quantity for a space curve when there is no canonical spanning surface whose area can be measured? A natural answer is to look at projections to 2D planes. Fix $\vec{a} \in S^2 \subset \mathbb{R}^3$ and project γ orthogonally onto $\vec{a}^\perp$. Then, for a closed curve $\gamma \in \mathcal{M}_{\text{id}}$, the signed area enclosed by the projection of γ to the plane $\vec{a}^\perp$ is given by

$$\begin{aligned} \mathcal{A}_{\vec{a}}(\gamma) &:= \frac{1}{2} \oint \langle \vec{a}, (\vec{\gamma} - \langle \vec{\gamma}, \vec{a} \rangle \vec{a}) \times (\vec{\gamma}' - \langle \vec{\gamma}', \vec{a} \rangle \vec{a}) \rangle ds \\ &= \frac{1}{2} \oint \langle \vec{a}, \vec{\gamma} \times \vec{\gamma}' \rangle ds \\ &= \left\langle \vec{a}, \frac{1}{2} \oint \vec{\gamma} \times \vec{\gamma}' ds \right\rangle, \end{aligned}$$

where the area form on $\vec{a}^\perp$ is given by $\langle \vec{a}, \cdot \times \cdot \rangle$. This motivates the definition of the *area vector*

$$\frac{1}{2} \oint \vec{\gamma} \times \vec{\gamma}' ds \in \mathbb{R}^3, \quad (7)$$

whose component $\langle \vec{a}, \frac{1}{2} \oint \vec{\gamma} \times \vec{\gamma}' ds \rangle$ is the curve's signed projected area along $\vec{a}$ (Fig. 7).

For a non-closed quasi-periodic curve, changing the fundamental domain produces a boundary term, so the line integral in Eq. (7) is not well defined by itself. Lemma 2.3 states that the relevant components of the *completed* area vector

$$\mathcal{A}(\gamma) := \frac{1}{2} \int_I \vec{\gamma} \times \vec{\gamma}' ds - \frac{1}{2} \vec{b} \times \vec{\gamma}(t_0) \in \mathbb{R}^3, \quad (8)$$

become independent of the chosen fundamental domain (Fig. 8).

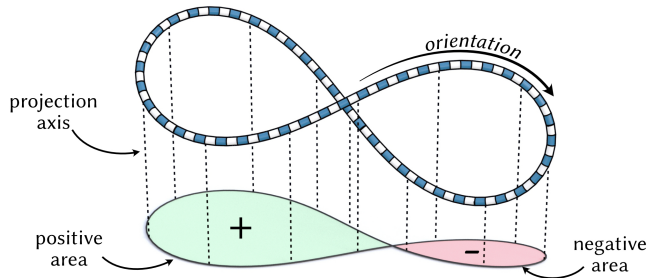


Fig. 7. The component of the area vector in the projection direction records the signed area enclosed by the projected curve, with sign determined by orientation.

Lemma 2.3. Let $\gamma \in \mathcal{M}_g$ have monodromy $g = (R, \vec{b})$, and let $\mathcal{A}(\gamma)$ be defined by Eq. (8) on a fundamental domain $I = [t_0, \tau(t_0)]$. Then the following area data are independent of the choice of I :

- (i) If $g = \text{id}$, the full vector $\mathcal{A}(\gamma)$.

³Pure half-turn monodromy has an additional disconnected centralizer component characterized by $Q\vec{s}_g = -\vec{s}_g$. We exclude this case throughout. Since $g^2 = \text{id}$, two consecutive fundamental domains form a closed curve, so it may instead be treated within the closed theory with an imposed half-turn symmetry.

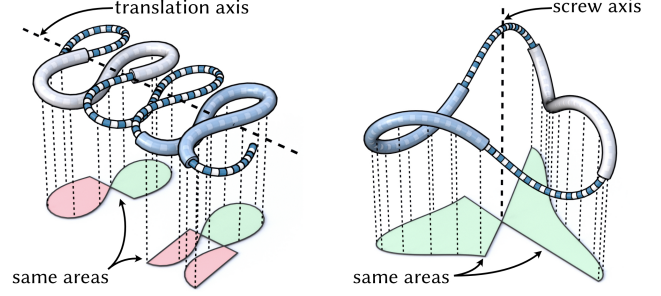


Fig. 8. The intrinsic components of the area vector depend on the monodromy. For pure translations the full area vector is well-defined, while for screw monodromy only the axial component is independent of the chosen fundamental domain.

- (ii) If $R = \text{Id}$ and $\vec{b} \neq 0$, the full vector $\mathcal{A}(\gamma)$.
 (iii) If $R \neq \text{Id}$, after choosing the origin on the screw axis $\vec{s}_g$ of g , the projection $\langle \vec{s}_g, \mathcal{A}(\gamma) \rangle$ onto the screw axis.

PROOF. See Appendix C. $\square$

2.3.3 Volume vector of a space curve. To define the analogous volume-type quantity, again start with a closed curve. Fix $\vec{a} \in S^2 \subset \mathbb{R}^3$ and consider the rotational vector field $X_{\vec{a}}(\vec{r}) = \vec{a} \times \vec{r}$. Since $\text{div } X_{\vec{a}} = 0$, the flux

$$\Upsilon_{\vec{a}}(\gamma) := \iint_{\Sigma} \langle X_{\vec{a}}(\vec{r}), N \rangle dS \quad (9)$$

through any surface Σ with boundary γ and surface unit normal N is independent of the choice of Σ . Evaluating this flux on the cone $\Sigma = \{u\vec{\gamma}(s) \mid u \in [0, 1], s \in I\}$ gives

$$\begin{aligned} \Upsilon_{\vec{a}}(\gamma) &= \int_0^1 \oint \langle \vec{a} \times (u\vec{\gamma}), \partial_u \vec{r} \times \partial_s \vec{r} \rangle ds du \\ &= \int_0^1 u^2 du \oint \langle \vec{a} \times \vec{\gamma}, \vec{\gamma} \times \vec{\gamma}' \rangle ds \\ &= \frac{1}{3} \oint \langle \vec{a} \times \vec{\gamma}, \vec{\gamma} \times \vec{\gamma}' \rangle ds \\ &= \left\langle \vec{a}, \frac{1}{3} \oint \vec{\gamma} \times (\vec{\gamma} \times \vec{\gamma}') ds \right\rangle. \end{aligned}$$

This motivates the definition of the *volume vector* (Fig. 9)

$$\frac{1}{3} \oint \vec{\gamma} \times (\vec{\gamma} \times \vec{\gamma}') ds \in \mathbb{R}^3. \quad (10)$$

For a closed curve, the component $\langle \vec{a}, \mathcal{V}(\gamma) \rangle$ is the flux of the rotational vector field $X_{\vec{a}}(\vec{r}) = \vec{a} \times \vec{r}$, which is proportional to the signed volume swept out by rotating a spanning surface around this axis.

Analogously, the volume vector for a non-closed quasi-periodic curve requires additional care: the cone has an additional boundary, so the line integral in Eq. (10) depends on the chosen fundamental domain. Lemma 2.4 states that the relevant components of the *completed* volume vector

$$\mathcal{V}(\gamma) := \frac{1}{3} \int_I \vec{\gamma} \times (\vec{\gamma} \times \vec{\gamma}') ds - \frac{1}{6} \vec{\gamma}(t_0) \times (\vec{b} \times \vec{\gamma}(t_0)) \in \mathbb{R}^3, \quad (11)$$

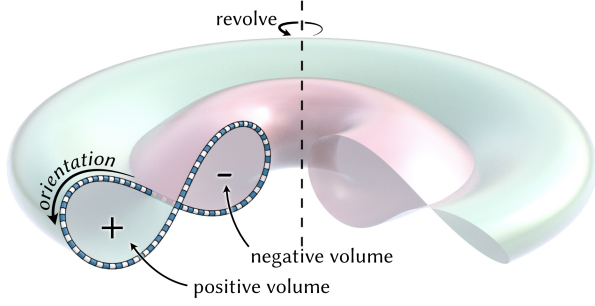


Fig. 9. The corresponding component of the volume vector measures the signed volume swept by rotating a spanning surface around the chosen axis.

become independent of the chosen fundamental domain $[t_0, \tau(t_0)]$.

Lemma 2.4. Let $\gamma \in \mathcal{M}_g$ have monodromy $g = (R, \vec{b})$, and let $\mathcal{V}(\gamma)$ be defined by Eq. (11) on a fundamental domain $I = [t_0, \tau(t_0)]$. Then the following volume data are independent of the choice of I :

- (i) If $g = \text{id}$, the full vector $\mathcal{V}(\gamma)$.
- (ii) If $R = \text{Id}$ and $\vec{b} \neq 0$, with $\vec{s}_g = \vec{b}/|\vec{b}|$, the projection $\langle \vec{s}_g, \mathcal{V}(\gamma) \rangle$ onto the chosen translation axis.
- (iii) If $R \neq \text{Id}$, after choosing the origin on the screw axis $\vec{s}_g$ of g , the projection $\langle \vec{s}_g, \mathcal{V}(\gamma) \rangle$ onto the screw axis.

PROOF. See Appendix D. $\square$

2.3.4 Isoperimetric characterization of quasi-periodic elastic curves. We now turn to the isoperimetric characterization of elastic curves for quasi-periodic curves. The constraints are the well-defined components of the area and volume vectors, which depend on the monodromy as established in Lemmas 2.3 and 2.4.

Theorem 2.5. Let $\gamma \in \mathcal{M}_g$ have monodromy $g = (R, \vec{b})$, and choose the origin on the screw axis whenever $R \neq \text{Id}$. For any $\vec{\lambda}_1, \vec{\lambda}_2 \in \mathbb{R}^3$, where $\vec{\lambda}_1$ is arbitrary if $g = \text{id}$ and parallel to $\vec{s}_g$ otherwise, while $\vec{\lambda}_2$ is arbitrary if $R = \text{Id}$ and parallel to $\vec{s}_g$ if $R \neq \text{Id}$, the following are equivalent:

- (i) The functional $\mathcal{L} + \langle \mathcal{V} | \vec{\lambda}_1 \rangle + \langle \mathcal{A} | \vec{\lambda}_2 \rangle$ is stationary at γ with respect to every variation $\dot{\gamma} \in T_\gamma \mathcal{M}_g$, i.e., $\dot{\gamma}(\tau(t)) = R\dot{\gamma}(t)$.
- (ii) The corresponding local first variation on the underlying infinite curve vanishes for every compactly supported variation $\dot{\gamma} \in C_0^\infty(\mathbb{R}, \mathbb{R}^3)$.

Moreover, γ is elastic if and only if there exists a pair $(\vec{\lambda}_1, \vec{\lambda}_2)$ satisfying the conditions above for which either, and hence both, of these equivalent conditions holds.

PROOF. See Appendix E. $\square$

Theorem 2.5 ensures that we do not need to distinguish between variations on the space of curves $\mathcal{M}$ under variations with compact support and on the space of quasi-periodic curves $\mathcal{M}_g$ under variations compatible with the quasi-periodicity.

Corollary 2.6 (Isoperimetric characterization of elastic curves). Let $\gamma \in \mathcal{M}_g$ have monodromy $g = (R, \vec{b}) \in \text{SE}(3)$, and choose the origin on the screw axis of g whenever $R \neq \text{Id}$. Then γ is elastic if and only if

it is a critical point of $\mathcal{L}$ at fixed well-defined components of the area and volume vectors.

PROOF. Constrained criticality is equivalent to stationarity of $\mathcal{L}$ plus a linear combination of the well-defined area–volume components, so the claim follows from Theorem 2.5. $\square$

While Corollary 2.6 was first observed by Chern et al. [2020, Cor. 2], we provide a proof of Theorem 2.5 (and thus Corollary 2.6) for the quasi-periodic case and using only elementary techniques from the calculus of variations.

3 Geometric Mechanics of Elastic Curves

The isoperimetric characterization of Section 2.3 has a deeper geometric explanation: the area and volume vectors are not arbitrary integral constraints, but the *momentum map* of a natural symmetry group action on the space of curves. Making this precise requires a brief review of the relevant *geometric mechanics*. For a more in-depth overview, we refer to [Marsden and Ratiu 1999]. We develop the theory directly for quasi-periodic curves $\gamma \in \mathcal{M}_g$, with the classical closed curve case $g = \text{id}$ a special case.

3.1 Symmetries of $\mathcal{M}_g$

A *Lie group* is a group that is simultaneously a smooth manifold, with group multiplication and inversion smooth maps. The tangent space at the identity of any Lie group G is its Lie algebra $\mathfrak{g} = T_{\text{id}}G$, which encodes the infinitesimal structure of the group. When a Lie group acts on a configuration space and the Lagrangian is invariant under this action, the group constitutes a symmetry of the variational problem, and Noether’s theorem associates a conserved quantity to each such symmetry.

3.1.1 Adjoint and Coadjoint actions. Every Lie group G acts on its Lie algebra $\mathfrak{g}$ by the *adjoint action*

$$\text{Ad}_g: \mathfrak{g} \rightarrow \mathfrak{g}, \quad \text{Ad}_g \xi = \left. \frac{d}{dt} \right|_{t=0} g h(t) g^{-1},$$

where $t \mapsto h(t) \in G$ is any curve with $h(0) = \text{id}$ and $h'(0) = \xi$. It describes how infinitesimal rigid body motions transform under conjugation by a finite group element. Dualizing this action gives the *coadjoint action*

$$\text{Ad}_g^*: \mathfrak{g}^* \rightarrow \mathfrak{g}^*, \quad g \in G,$$

defined by $\langle \text{Ad}_g^* \eta | x \rangle = \langle \eta | \text{Ad}_{g^{-1}} x \rangle$ for $x \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$, where $\langle \cdot | \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ is the natural pairing. Intuitively, Ad_g^* describes how *momenta*—physical quantities dual to infinitesimal rigid body motions—transform under a finite rigid motion g .

3.1.2 The special Euclidean group. For our purposes, the *special Euclidean group* $\text{SE}(3)$ of orientation-preserving rigid motions of $\mathbb{R}^3$ is the primary example. It is a six-dimensional Lie group whose Lie algebra $\mathfrak{se}(3)$ consists of infinitesimal rigid body motions, identified with pairs $(\vec{\omega}, \vec{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$, where $\vec{\omega} \in \mathbb{R}^3$ is an infinitesimal rotation and $\vec{v} \in \mathbb{R}^3$ an infinitesimal translation. Elements $(\vec{l}, \vec{p}) \in \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathfrak{se}(3)^*$ of the dual space canonically pair with elements $(\vec{\omega}, \vec{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathfrak{se}(3)$ by

$$\langle (\vec{l}, \vec{p}) | (\vec{\omega}, \vec{v}) \rangle = \langle \vec{l} | \vec{\omega} \rangle + \langle \vec{p} | \vec{v} \rangle.$$

The coadjoint action of $SE(3)$ on its dual Lie algebra $\mathfrak{se}(3)^*$ is given by

$$\text{Ad}_{(R, \vec{b})}^* (\vec{l}, \vec{p}) = (R\vec{l} + \vec{b} \times R\vec{p}, R\vec{p}), \quad (12)$$

where $(R, \vec{b}) \in SE(3)$ and $(\vec{l}, \vec{p}) \in \mathfrak{se}(3)^*$.

The Lie algebra of the centralizer subgroups $Z(g)$ of elements $g \in SE(3)$ as in Section 2.3.1 is denoted by $\mathfrak{z}(g)$ and given by the corresponding subspaces

(i) If $g = \text{id}$, then $\mathfrak{z}(g) = \mathfrak{se}(3)$.

(ii) If $R = \text{Id}$, $\vec{b} \neq 0$,

$$\mathfrak{z}(g) = \{(\vec{\omega}, \vec{v}) \mid \vec{\omega}, \vec{v} \in \mathbb{R}^3, \vec{\omega} \parallel \vec{s}_g\}.$$

(iii) If $g = (R, \vec{b})$ with $R \neq \text{Id}$,

$$\mathfrak{z}(g) = \{(\vec{\omega}, \vec{v}) \mid \vec{\omega}, \vec{v} \in \mathbb{R}^3, \vec{\omega}, \vec{v} \parallel \vec{s}_g\}.$$

3.1.3 Covariance of area and volume under $SE(3)$. Although the components identified above are independent of the chosen fundamental domain, they are not invariant under arbitrary rigid motions of the curve in the ambient space. Instead, area and volume transform covariantly with the placement of the curve and with the chosen screw-axis gauge.

Theorem 3.1. *Let $\gamma \in \mathcal{M}_g$ with monodromy $g = (R, \vec{b}) \in SE(3)$, and let $h = (Q, \vec{c}) \in Z(g)$ be a symmetry of $\mathcal{M}_g$. Then:*

(i) *If $g = \text{id}$, any $h \in SE(3)$ is allowed and*

$$\begin{aligned} \mathcal{A}(h \circ \gamma) &= Q\mathcal{A}(\gamma), \\ \mathcal{V}(h \circ \gamma) &= Q\mathcal{V}(\gamma) + \vec{c} \times Q\mathcal{A}(\gamma). \end{aligned}$$

(ii) *If $R = \text{Id}$ and $\vec{b} \neq 0$, then $\vec{s}_g = \vec{b}/|\vec{b}|$, and $h = (Q, \vec{c}) \in Z(g)$ has $Q\vec{s}_g = \vec{s}_g$ with $\vec{c} \in \mathbb{R}^3$ arbitrary. Writing $\vec{c}^\perp = \vec{c} - \langle \vec{s}_g, \vec{c} \rangle \vec{s}_g$,*

$$\begin{aligned} \mathcal{A}(h \circ \gamma) &= Q\mathcal{A}(\gamma) + \vec{c} \times \vec{b}, \\ \langle \vec{s}_g, \mathcal{V}(h \circ \gamma) \rangle &= \langle \vec{s}_g, Q\mathcal{V}(\gamma) + \vec{c} \times Q\mathcal{A}(\gamma) \rangle - \frac{1}{2} |\vec{b}| |\vec{c}^\perp|^2. \end{aligned}$$

When the translation preserves the axis $\vec{s}_g$ (i.e., $\vec{c} \parallel \vec{s}_g$) the two additional terms vanish.

(iii) *If $R \neq \text{Id}$, h must be a screw motion about the axis $\mathbb{R}\vec{s}_g$, and the axial projections transform as*

$$\begin{aligned} \langle \vec{s}_g, \mathcal{A}(h \circ \gamma) \rangle &= \langle \vec{s}_g, Q\mathcal{A}(\gamma) \rangle, \\ \langle \vec{s}_g, \mathcal{V}(h \circ \gamma) \rangle &= \langle \vec{s}_g, Q\mathcal{V}(\gamma) + \vec{c} \times Q\mathcal{A}(\gamma) \rangle. \end{aligned}$$

PROOF. See Appendix F. $\square$

These transformation laws are the geometric heart of the paper and drive both the continuous and discrete theories that follow. In Section 3.3 we identify the restriction of $(\mathcal{V}, \mathcal{A})$ to the compatible infinitesimal rigid motions as *momenta*, i.e., elements of $\mathfrak{z}(g)^*$ transforming under $Z(g)$, which explains why fixing them is the natural constraint for elastic curves.

3.2 $\mathcal{M}_g$ as a (pre-)symplectic manifold

In geometric mechanics, a *pre-symplectic structure* on a manifold is a closed 2-form. It turns out that the space $\mathcal{M}_g$ naturally carries such a structure. The *Marsden–Weinstein form* is given by

$$\Omega_\gamma^{MW}(\dot{\gamma}, \dot{\gamma}) = \int_I \det(\vec{\gamma}, \dot{\gamma}, d\vec{\gamma}), \quad (13)$$

for all $\dot{\gamma}, \dot{\gamma} \in T_\gamma \mathcal{M}_g$ [Marsden and Weinstein 1983].

Theorem 3.2. *The Marsden–Weinstein form Ω^{MW} is a pre-symplectic form on $\mathcal{M}_g$, i.e., a closed 2-form. Moreover, its kernel consists precisely of the reparametrization directions.*

PROOF. See Appendix G. $\square$

Geometrically, the Marsden–Weinstein form measures the signed volume swept along γ by the infinitesimal oriented area element spanned by $\dot{\gamma}$ and $\dot{\gamma}$.

This geometric interpretation also explains the degeneracy. A reparametrization direction is collinear with the tangent $\gamma' = d\gamma(\frac{\partial}{\partial s})$, the direction along which the prism is swept, so the prism collapses and has no volume. Thus, after quotienting by reparametrizations, the Marsden–Weinstein form descends to a closed, non-degenerate 2-form, i.e., a genuine *symplectic form*.

The action of the centralizer $Z(g)$ preserves the Marsden–Weinstein form on $\mathcal{M}_g$, i.e., this construction is compatible with the symmetries of $\mathcal{M}_g$.

Lemma 3.3. *The centralizer $Z(g)$ preserves the pre-symplectic form Ω^{MW} on $\mathcal{M}_g$. Consequently, the induced action of $Z(g)$ on the quotient of $\mathcal{M}_g$ by reparametrizations is symplectic.*

PROOF. See Appendix H. $\square$

3.3 The Momentum Map and Noether charges

Whenever a Lie group G acts on a symplectic manifold (M, Ω) by symplectomorphisms, one seeks a *momentum map*, i.e., a map

$$\mu: M \rightarrow \mathfrak{g}^*$$

satisfying, for every $\xi \in \mathfrak{g}$,

$$\iota_{X_\xi} \Omega = d\langle \mu | \xi \rangle, \quad (14)$$

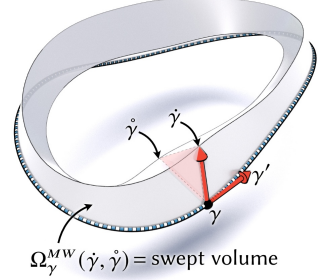
where X_ξ is the vector field on M generated by ξ . Thus each component $\langle \mu | \xi \rangle$ records the conserved quantity associated with the infinitesimal symmetry ξ , and the momentum map packages these *Noether charges* into a single $\mathfrak{g}^*$ -valued function. The same definition applies to the pre-symplectic form on $\mathcal{M}_g$, before passing to the quotient by reparametrizations.

The purpose of this section is to identify the area-volume data of Theorem 2.5 as momenta. Through this identification, the constraints in the isoperimetric problem are no longer external choices: they are the Noether charges of the rigid-motion symmetries that remain compatible with the monodromy.

3.3.1 The momentum map. The relevant symmetry group of $\mathcal{M}_g$ is the centralizer $Z(g)$, whose Lie algebra $\mathfrak{z}(g) \subset \mathfrak{se}(3)$ consists of the infinitesimal rigid motions compatible with the monodromy. Hence the momentum map records only the components of the area-volume pair visible to $\mathfrak{z}(g)$.

For $(\vec{\omega}, \vec{v}) \in \mathfrak{z}(g)$, the infinitesimal vector field on $\mathcal{M}_g$ is

$$X_{(\vec{\omega}, \vec{v})}(\gamma) = \vec{\omega} \times \vec{\gamma} + \vec{v}.$$



By Lemma 3.3, this action preserves Ω^{MW} , and its momentum map is as follows [Marsden and Ratiu 1999].

Theorem 3.4. *Let $g \in \text{SE}(3)$, and let $\iota_3: \mathfrak{z}(g) \hookrightarrow \mathfrak{se}(3)$ denote the inclusion of the Lie algebra of its centralizer. The momentum map for the $Z(g)$ -action on $(\mathcal{M}_g, \Omega^{MW})$ is*

$$\mu_g: \mathcal{M}_g \rightarrow \mathfrak{z}(g)^*, \gamma \mapsto \iota_3^* \begin{pmatrix} \mathcal{V}(\gamma) \\ \mathcal{A}(\gamma) \end{pmatrix}.$$

PROOF. See Appendix I. $\square$

Theorem 3.4 states that the momentum map on $\mathcal{M}_g$ is obtained from the full area-volume pair by restricting the covector $(\mathcal{V}, \mathcal{A}) \in \mathfrak{se}(3)^*$ to $\mathfrak{z}(g) \subset \mathfrak{se}(3)$. Thus the remaining Noether charges are:

(i) For $g = \text{id}$,

$$\mu_g(\gamma) = \begin{pmatrix} \mathcal{V}(\gamma) \\ \mathcal{A}(\gamma) \end{pmatrix},$$

which recovers the known angular and linear momenta of the Marsden–Weinstein form on the space of closed space curves (cf. [Shashikanth and Marsden 2003]).

(ii) For pure translational monodromy, $R = \text{Id}$ and $\vec{b} \neq 0$,

$$\mu_g(\gamma) = \begin{pmatrix} \langle \vec{b}/|\vec{b}|, \mathcal{V}(\gamma) \rangle \\ \mathcal{A}(\gamma) \end{pmatrix}.$$

(iii) For screw monodromy, $R \neq \text{Id}$,

$$\mu_g(\gamma) = \begin{pmatrix} \langle \vec{s}_g, \mathcal{V}(\gamma) \rangle \\ \langle \vec{s}_g, \mathcal{A}(\gamma) \rangle \end{pmatrix},$$

where $\vec{s}_g \in S^2$ is chosen such that $\mathbb{R}\vec{s}_g$ is the screw axis.

With the momentum map in hand, the otherwise unexpected area-volume constraints cease to be mysterious. On the fixed-monodromy space $\mathcal{M}_g$, the length $\mathcal{L}$ is invariant under the symmetry group $Z(g)$, and μ_g records the corresponding Noether charges. Fixing the area-volume components selected by the monodromy is therefore equivalent to fixing $\eta \in \mathfrak{z}(g)^*$. The admissible curves form the momentum level set

$$\mu_g^{-1}(\eta) = \{\gamma \in \mathcal{M}_g \mid \mu_g(\gamma) = \eta\}.$$

Equivalently, the constrained variational problem is to find critical points of

$$\mathcal{L}|_{\mu_g^{-1}(\eta)}: \mu_g^{-1}(\eta) \rightarrow \mathbb{R}.$$

Thus the monodromy selects the constraints by selecting the rigid-motion symmetries that survive.

4 Discretization

We now transfer the continuous construction to discrete polygonal curves. Computationally, this is the central simplification: once length, area, and volume are defined on a polygon, a discrete elastica is obtained by constrained optimization over its vertex positions. No curvature model or curve-flow PDE must be discretized or integrated. The guiding structural requirement is that the discrete area and volume vectors obey the same transformation laws as their continuous counterparts.

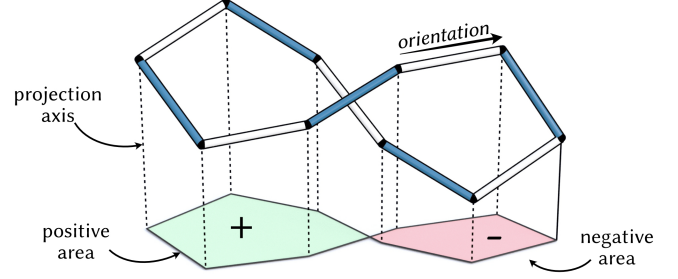


Fig. 10. Evaluating the projected-area formula on the piecewise-linear interpolant gives a discrete area vector that satisfies the same rigid-motion transformation law as in the smooth theory.

4.1 Discrete quasi-periodic curves

Fix $3 \leq N \in \mathbb{N}$ and let $I = \{1, \dots, N\} \subset \mathbb{Z}$ be the fundamental index interval. A discrete curve is a map $\gamma: \mathbb{Z} \rightarrow \mathbb{R}^3$, $i \mapsto \vec{\gamma}_i$. It is *quasi-periodic* with monodromy $g = (R, \vec{b}) \in \text{SE}(3)$ if for all $i \in \mathbb{Z}$

$$\vec{\gamma}_{i+N} = g(\vec{\gamma}_i) = R\vec{\gamma}_i + \vec{b}.$$

Such a curve is determined by its values on I , namely its N vertices $(\vec{\gamma}_1, \dots, \vec{\gamma}_N)$ of the fundamental domain. Closed polygons correspond to $g = \text{id}$. We write

$$\mathcal{M}_g = \{\gamma: \mathbb{Z} \rightarrow \mathbb{R}^3 \mid \vec{\gamma}_{i+N} = g(\vec{\gamma}_i) \forall i \in \mathbb{Z}\}$$

for the space of discrete quasi-periodic curves⁴ with monodromy g .

Piecewise-linear realization. We denote by $\mathcal{I}$ the map sending each $\gamma \in \mathcal{M}_g$ to its continuous piecewise-linear quasi-periodic interpolant. On the i -th edge,

$$\vec{\gamma}_i(s) = (1-s)\vec{\gamma}_i + s\vec{\gamma}_{i+1}, \quad s \in [0, 1].$$

Writing $\vec{e}_i = \vec{\gamma}_{i+1} - \vec{\gamma}_i$, we have $d\vec{\gamma} = \vec{e}_i ds$ on this edge. Vector fields along γ , such as variations, are interpolated in the same way. For $X \in T_\gamma \mathcal{M}_g$, set

$$X_i(s) := (1-s)X_i + sX_{i+1}, \quad s \in [0, 1].$$

These interpolants are continuous and piecewise smooth, with possible loss of smoothness only at the vertices. Evaluating the integral formulas for $\mathcal{L}$, $\mathcal{A}$, $\mathcal{V}$, and Ω^{MW} edgewise gives well-defined extensions to continuous piecewise-smooth quasi-periodic curves, for which we use the same notation.

4.2 Discrete functionals

The length of a discrete polygonal curve is given by

$$\mathcal{L}(\gamma) = \sum_{i=1}^N |\vec{\gamma}_{i+1} - \vec{\gamma}_i|. \quad (15)$$

The discrete area and volume vectors are obtained by evaluating the continuous formulas on the piecewise-linear interpolant (Figs. 10 and 11). On edge i , the area contribution is

$$\frac{1}{2} \int_0^1 \vec{\gamma}_i(s) \times \vec{e}_i ds = \frac{1}{2} \vec{\gamma}_i \times \vec{\gamma}_{i+1}.$$

⁴For the differentiable and Hamiltonian constructions below, we restrict to polygons with $\vec{\gamma}_{i+1} \neq \vec{\gamma}_i$ for every i , retaining the notation $\mathcal{M}_g$ for this open subset.

Summing over the fundamental domain gives

$$A(\gamma) = \frac{1}{2} \sum_{i=1}^N \vec{\gamma}_i \times \vec{\gamma}_{i+1} - \frac{1}{2} \vec{b} \times \vec{\gamma}_1. \quad (16)$$

Similarly, exact evaluation of the continuous volume integral on each linear edge gives

$$V(\gamma) = \frac{1}{3} \sum_{i=1}^N \frac{\vec{\gamma}_i + \vec{\gamma}_{i+1}}{2} \times (\vec{\gamma}_i \times \vec{\gamma}_{i+1}) - \frac{1}{6} \vec{\gamma}_1 \times (\vec{b} \times \vec{\gamma}_1). \quad (17)$$

By construction, $A = I^* \mathcal{A}$ and $V = I^* \mathcal{V}$, where the continuous vectors are understood in the edgewise sense above. Their well-definedness and transformation laws therefore transfer verbatim from the continuous theory (Lemmas 2.3 and 2.4 and Theorem 3.1), giving the following discrete counterpart.

Corollary 4.1. *The monodromy-compatible components of A and V are independent of the chosen discrete fundamental domain and transform exactly by the laws of Theorem 3.1. In particular, for closed polygons the full pair (V, A) transforms by the coadjoint action of $SE(3)$, whereas for non-closed polygons the monodromy selects the same components as in the smooth theory.*

These constraints on discrete curves enter the momentum map below. Restricting (V, A) to $\mathfrak{z}(g)$ gives a covector in $\mathfrak{z}(g)^*$.

4.3 The discrete Marsden–Weinstein form

The Marsden–Weinstein form also has a direct discrete counterpart. Using the edgewise extension described above, we define $\Omega^{MW} := I^* \Omega^{MW}$, where the differential of I interpolates variations in the same piecewise-linear basis. The proof of Theorem 3.2 applies edgewise, with the boundary terms at interior vertices canceling, so this extension remains closed. Since the exterior derivative commutes with pullback,

$$d\Omega^{MW} = d(I^* \Omega^{MW}) = I^*(d\Omega^{MW}) = 0,$$

where we use that Ω^{MW} is closed by Theorem 3.2. Moreover, the finite-sum formula makes bilinearity and skew-symmetry immediate, so that we conclude:

Proposition 4.2. *The discrete Marsden–Weinstein form Ω^{MW} is a pre-symplectic 2-form on M_g .*

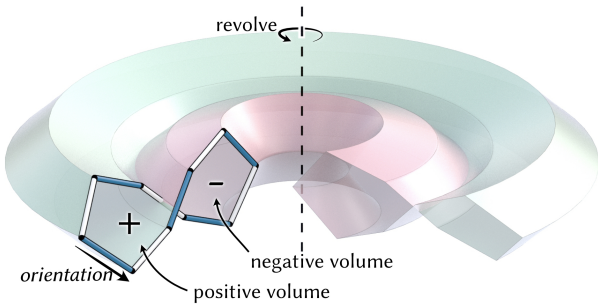
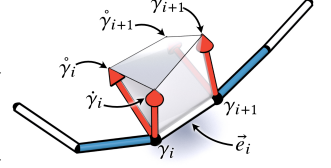


Fig. 11. Exact integration along polygon edges gives a discrete volume vector whose transformation law agrees with the coadjoint action of rigid motions.

Geometrically, along each edge the linearly interpolated variations $\dot{\gamma}$ and $\dot{\gamma}$ span an infinitesimal oriented area element. Its sweep along the edge determines an infinitesimal signed volume, and Eq. (18) sums these edgewise contributions (see inset).



Matrix representation. For computations, we assemble an explicit matrix representation Ω of Ω^{MW} . For variations $\dot{\gamma}, \dot{\gamma} \in T_\gamma M_g$, piecewise-linear interpolation gives

$$\begin{aligned} \Omega_\gamma^{MW}(\dot{\gamma}, \dot{\gamma}) &= \sum_{i=1}^N \int_0^1 \det(\dot{\gamma}_i(s), \dot{\gamma}_i(s), \vec{e}_i) ds \\ &= \sum_{i=1}^N \left[\frac{1}{3} \det(\dot{\gamma}_i, \dot{\gamma}_i, \vec{e}_i) + \frac{1}{6} \det(\dot{\gamma}_{i+1}, \dot{\gamma}_i, \vec{e}_i) \right. \\ &\quad \left. + \frac{1}{6} \det(\dot{\gamma}_i, \dot{\gamma}_{i+1}, \vec{e}_i) + \frac{1}{3} \det(\dot{\gamma}_{i+1}, \dot{\gamma}_{i+1}, \vec{e}_i) \right]. \end{aligned} \quad (18)$$

Thus, Ω can be assembled directly from the vertex stencil without further integration.

Writing $[\vec{e}] : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ for the skew map $[\vec{e}] \vec{x} = \vec{e} \times \vec{x}$, and using the convention $\Omega_\gamma^{MW}(\dot{\gamma}, \dot{\gamma}) = \dot{\gamma}^\top \Omega \dot{\gamma}$, the $3N \times 3N$ matrix Ω is block tridiagonal with wrap-around blocks. Away from the boundary, its three nonzero blocks in row i are

$$\Omega_{i,i-1} = \frac{1}{6} [\vec{e}_{i-1}], \quad \Omega_{i,i} = \frac{1}{3} [\vec{e}_{i-1} + \vec{e}_i], \quad \Omega_{i,i+1} = \frac{1}{6} [\vec{e}_i].$$

The diagonal block collects the two edges incident to vertex i , while each off-diagonal block comes from the shared edge. At the boundary, the same local stencil is evaluated on the padded vertices, using cyclic copies for closed curves and monodromy copies for quasi-periodic curves, as described in Section 5.1. Since Eq. (18) is alternating in its two variations, the assembled matrix Ω is skew-symmetric.

4.4 The discrete momentum map

The centralizer $Z(g)$ acts on M_g vertexwise, which by construction preserves Ω^{MW} . The momentum map of this action mirrors its continuous counterpart in Theorem 3.4.

Theorem 4.3. *Let $g \in SE(3)$, and let $i_3 : \mathfrak{z}(g) \hookrightarrow \mathfrak{se}(3)$ denote the inclusion of the Lie algebra of its centralizer. The discrete momentum map for the $Z(g)$ -action on (M_g, Ω^{MW}) is*

$$\mu_g^{\text{disc}} : M_g \rightarrow \mathfrak{z}(g)^*, \quad \gamma \mapsto i_3^* \begin{pmatrix} V(\gamma) \\ A(\gamma) \end{pmatrix}.$$

PROOF. See Appendix J. □

Unpacking the restriction to $\mathfrak{z}(g)$ yields the discrete analogues of the three cases in the continuous theory:

(i) For $g = \text{id}$,

$$\mu_g^{\text{disc}}(\gamma) = \begin{pmatrix} V(\gamma) \\ A(\gamma) \end{pmatrix}.$$

(ii) For pure translational monodromy, $R = \text{Id}$ and $\vec{b} \neq 0$,

$$\mu_g^{\text{disc}}(\gamma) = \begin{pmatrix} \langle \vec{b}/|\vec{b}|, V(\gamma) \rangle \\ A(\gamma) \end{pmatrix}.$$

(iii) For screw monodromy, $R \neq \text{Id}$,

$$\mu_g^{\text{disc}}(\gamma) = \begin{pmatrix} \langle \vec{s}_g, V(\gamma) \rangle \\ \langle \vec{s}_g, A(\gamma) \rangle \end{pmatrix},$$

where $\vec{s}_g \in S^2$ is chosen such that $\mathbb{R}\vec{s}_g$ is the screw axis.

The discrete area and volume components are therefore the Noether charges of the Hamiltonian $Z(g)$ -action on the pre-symplectic manifold $(M_g, \Omega^{\text{MW}})$.

4.5 Discrete Elastic Curves

Finally, building on the discrete structural properties established above, we define our notion of discrete elastic curves, motivated by the continuous isoperimetric Theorem 2.5.

In the smooth theory, $\mathcal{L}$, $\mathcal{A}$, and $\mathcal{V}$ are invariant under reparametrization. An N -vertex polygon has no analogous continuous freedom, since resampling generally changes its geometry. We therefore fix the discrete arc-length gauge by requiring equal edge lengths and denote the resulting space of quasi-periodic polygons with monodromy $g \in \text{SE}(3)$ by

$$M_g^{\text{arc}} := \{\gamma \in M_g \mid |\vec{e}_i| = |\vec{e}_j|, \forall i, j \in I\}.$$

Definition 4.4. A curve $\gamma \in M_g^{\text{arc}}$ is discrete elastic if it is a critical point of the discrete length functional L for fixed discrete momentum data $\mu_g^{\text{disc}}(\gamma) \in \mathfrak{z}(g)^*$.

Therefore, in the computations below, we seek discrete elastica that are constrained local minimizers of the *discrete isoperimetric problem*

$$\begin{cases} \min_{\gamma \in M_g^{\text{arc}}} L(\gamma), \\ \text{s.t. } \mu_g^{\text{disc}}(\gamma) = \mu_0. \end{cases} \quad (19)$$

Here μ_g^{disc} is the discrete momentum map of Theorem 4.3, so that the momentum constraint in Eq. (19) fixes exactly those components of (V, A) that are compatible with the monodromy g .

We thus obtain an explicit finite-dimensional variational problem posed directly on polygon vertices and whose solutions exhibit consistent geometries across different resolutions (Figs. 1, 12, and 21). This formulation only involves the geometric quantities L , A , and V that can be evaluated directly on discrete polygonal curves and no further auxiliary discretization is required.

5 Numerical Optimization

The discrete isoperimetric problem (19) is a low-order, finite-dimensional constrained problem with the polygon vertices as variables and can therefore be solved directly using standard optimization methods.

5.1 Quasi-periodicity

Quasi-periodicity can be imposed in two ways, akin to the two characterizations in Theorem 2.5. One restricts to quasi-periodic curves in M_g and enforces the monodromy as a boundary condition. The other, which we use, treats the curve as de facto infinite and generates whatever lies outside the fundamental domain from the monodromy. Either way only the N vertices $\vec{\gamma}_1, \dots, \vec{\gamma}_N$ of one fundamental domain are optimized, and curves of arbitrary length can be obtained from the monodromy relation at no additional cost.

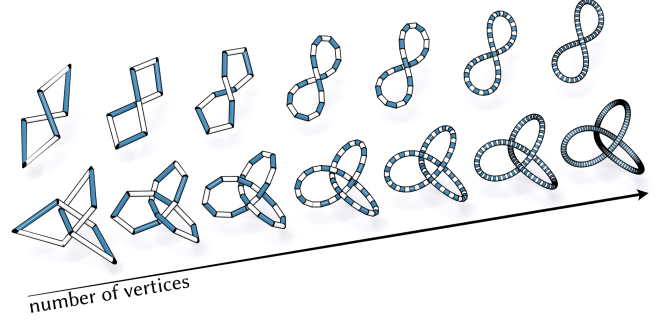


Fig. 12. The two rows show polygonal length minimizers at prescribed area and volume for two closed curves, with the number of vertices increasing from left to right. Even at the coarsest resolutions, the solutions faithfully reproduce the distinctive shapes of the corresponding smooth elastica, while refinement progressively improves the agreement.

Concretely, whenever a stencil reaches outside the fundamental domain, the required vertices are generated from the monodromy,

$$\vec{\gamma}_{N+1} = g(\vec{\gamma}_1), \quad \vec{\gamma}_0 = g^{-1}(\vec{\gamma}_N).$$

Since the functionals used here depend only on adjacent edges, one ghost vertex on each side contains all local information from the infinite curve. We evaluate the functionals on the padded curve and pull their gradients back through the padding map to the N fundamental-domain variables. Thus the monodromy is handled entirely by padding, while the optimizer sees only the fundamental variables.

5.2 Constraints

To identify solutions to Eq. (19), we solve the constrained optimization problem

$$\begin{cases} \min_{\gamma \in M_g} L(\gamma), \\ \text{s.t. } C_g(\gamma) = 0. \end{cases} \quad (20)$$

Here $C_g: M_g \rightarrow \mathbb{R}^m$ is the constraint map, comprised of momentum and equal-edge constraints. Writing $\ell(\gamma) := \frac{1}{N}L(\gamma)$, we set⁵

$$C_g(\gamma) = \begin{pmatrix} \mu_g^{\text{disc}}(\gamma) - \mu_0 \\ |\vec{e}_1| - \ell(\gamma) \\ \vdots \\ |\vec{e}_N| - \ell(\gamma) \end{pmatrix}. \quad (21)$$

Thus the computed stationary points are discrete arc-length parametrized curves satisfying the constraints on the components of (V, A) compatible with the monodromy g .

Remark 5.1. The equal-edge constraints fix the discrete arc-length gauge but leave residual symmetries: cyclic relabeling for closed polygons and rigid motions $h \in Z(g)$ preserving the prescribed momentum data. These leave the geometry unchanged but may appear numerically as null or near-null directions.

⁵The equal-edge residuals sum identically to zero, so only $N - 1$ are independent. We retain all N to preserve cyclic symmetry. This leaves only an additive indeterminacy in the associated multipliers and does not affect the primal problem.

5.3 Optimization

The input is a monodromy $g \in \text{SE}(3)$, an initial polygon $\gamma^0 \in M_g$, and prescribed momentum data $\mu_0 \in \mathfrak{z}(g)^*$. The output is a polygonal curve approximating a stationary point of discrete length at fixed discrete momentum.

We solve Eq. (20) by a standard augmented Lagrangian method. For multipliers $\lambda \in \mathbb{R}^m$ and penalty parameter $\rho > 0$, define

$$\mathcal{L}_\rho(\gamma, \lambda) = L(\gamma) + \langle \lambda | C_g(\gamma) \rangle + \frac{\rho}{2} \|C_g(\gamma)\|^2. \quad (22)$$

The resulting iteration is summarized in Alg. 1. Each inner problem is solved by a quasi-Newton method using the analytic gradient

$$\nabla_\gamma \mathcal{L}_\rho = \nabla_\gamma L + dC_g(\gamma)^\top (\lambda + \rho C_g(\gamma)). \quad (23)$$

The low differential order also keeps the reduced Hessian well conditioned under refinement, which is what makes a quasi-Newton method practical here (Fig. 13).

The length and equal-edge terms have explicit local derivatives, while A and V are polynomial in the vertex positions. Function and gradient evaluations can therefore be assembled edgewise at linear cost in N . The same routine applies to every resolution and monodromy type, with g entering only through the padded boundary data and the momentum components selected by C_g .

Algorithm 1 DiscreteIsoperimetricProblem (g, γ^0, μ_0)

Input: $g \in \text{SE}(3)$, $\gamma^0 \in M_g$, $\mu_0 \in \mathfrak{z}(g)^*$

Output: discrete elastic $\gamma \in M_g^{\text{arc}}$ with $\mu_g^{\text{disc}}(\gamma) = \mu_0$

```

1:  $\gamma \leftarrow \gamma^0$ 
2: initialize  $\lambda$  and  $\rho > 0$ 
3: while not converged do
4:    $\hat{\gamma} \leftarrow \text{PAD}(\gamma, g)$  ▷ Section 5.1
5:    $C_g \leftarrow \text{CONSTRAINTMAP}(g, \mu_0)$  ▷ Eq. (21)
6:    $\hat{\gamma}|_I \leftarrow \text{argmin } \mathcal{L}_\rho(\cdot | I, \lambda)$  ▷ Eqs. (22) and (23)
7:    $\gamma \leftarrow \text{DISCARDADDING}(\hat{\gamma})$ 
8:    $\lambda \leftarrow \lambda + \rho C_g(\gamma)$ 
9:   if the constraint residual stalls then
10:     increase  $\rho$ 
11: return  $\gamma$ 
```

Remark 5.2. In practice, a useful warm start is obtained by first minimizing the discrete Dirichlet energy $E(\gamma) = \frac{1}{2} \sum_{i=1}^N |\tilde{e}_i|^2$ subject only to the momentum constraints. After approximate convergence, we impose the equal-edge constraints and solve the original length gauge problem in the discrete arc-length gauge.

6 Hamiltonian Flows on Space Curves

The discrete Marsden–Weinstein form of Section 4.3 not only identifies the isoperimetric constraints as momentum data but also provides, as in the smooth theory, the phase-space structure underlying Hamiltonian evolutions of curves. In this section, we apply this construction directly to polygonal curves, focusing on discrete analogues of tangential reparametrization, vortex-filament motion, and the modified Korteweg–de Vries flow. While the framework is more general, these examples are particularly natural because their Hamiltonians can be evaluated directly on polygons.

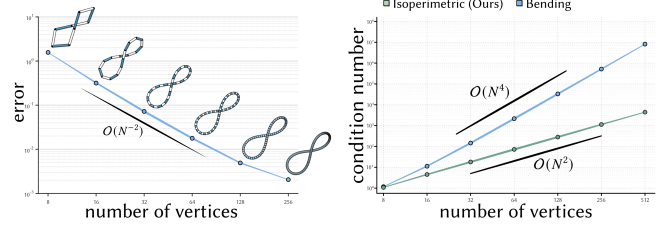


Fig. 13. *Left:* Our discrete elastica converge quadratically to the smooth reference solution under refinement. *Right:* The reduced Hessian of our isoperimetric formulation stays far better conditioned, growing like N^2 against N^4 for the bending-energy baseline, as their second- and fourth-order stationarity conditions predict.

6.1 Spectrum of Marsden–Weinstein form

In the continuous setup, the Marsden–Weinstein form is a pre-symplectic closed 2-form whose only degeneracy is given by reparametrization of curves. To investigate how the properties of the Marsden–Weinstein form carry over to the discrete setting, we examine the spectrum of its associated linear operator.

6.1.1 Spectrum of the continuous Marsden–Weinstein form. Since Ω_Y^{MW} is a 2-form rather than an operator, we define its spectrum through its L^2 -Riesz representative J_Y . For $X, Y \in T_Y M_g$ it is given by

$$\Omega_Y^{\text{MW}}(X, Y) = \langle J_Y X, Y \rangle_{L^2}.$$

Eq. (13) yields

$$J_Y X = T \times X,$$

where $T = \gamma'$, so J_Y is pointwise cross product with the unit tangent.

The kernel of J_Y consists precisely of the tangential fields $X = fT$, where $f: \mathbb{R} \rightarrow \mathbb{R}$ is smooth and τ -periodic. These fields generate infinitesimal reparametrizations. On normal fields,

$$J_Y^2 = -\text{id}.$$

Hence J_Y induces a complex structure on $T_Y M_g / \ker J_Y$, naturally identified with the normal bundle [Chern et al. 2020]. This is the local linear model of the reduced Marsden–Weinstein form.

Together, these identities show that the spectrum of J_Y on vector fields along γ is

$$\sigma(J_Y) = \{0, \pm i\}.$$

The zero eigenspace corresponds to tangential directions, while the eigenvalues $\pm i$ are those of the 90° -rotation induced by $J_Y X = T \times X$ on each normal plane.

6.1.2 Spectrum of the discrete Marsden–Weinstein form. To compare the discrete form with the smooth operator J_Y , let Ω be the matrix assembled from Eq. (18) and equip the vertex fields with the finite-element L^2 metric represented by the mass matrix

$$(M_Y X)_i = \frac{|\tilde{e}_{i-1}|}{6} X_{i-1} + \frac{|\tilde{e}_{i-1}| + |\tilde{e}_i|}{3} X_i + \frac{|\tilde{e}_i|}{6} X_{i+1}. \quad (24)$$

The corresponding mass representative J_Y is defined by

$$\Omega_Y^{\text{MW}}(X, Y) = \langle J_Y X, Y \rangle_{M_Y}.$$

H	dH_Y	X_H	Interpretation
$\langle \vec{v}, \mathcal{V} \rangle$	$\vec{y}' \times (\vec{v} \times \vec{y})$	$X_{\langle \vec{v}, \mathcal{V} \rangle} = \vec{v} \times \vec{y}$	rotation about $\vec{v}$
$\langle \vec{v}, \mathcal{A} \rangle$	$\vec{y}' \times \vec{v}$	$X_{\langle \vec{v}, \mathcal{A} \rangle} = \vec{v}$	translation along $\vec{v}$
0	0	$X_0 = \vec{y}'$	tangent flow
$\mathcal{L}$	$-\vec{y}''$	$X_{\mathcal{L}} = \vec{y}' \times \vec{y}''$	vortex-filament flow
Φ	$-\vec{y}' \times \vec{y}'''$	$X_{\Phi} = -\vec{y}''' - \frac{3}{2} \vec{y}'' ^2 \vec{y}'$	mKdV flow
$\mathcal{B}$	$(\vec{y}''' + \frac{3}{2} \vec{y}'' ^2 \vec{y}')'$	$X_{\mathcal{B}} = -\vec{y}' \times \vec{y}''' - \frac{3}{2} \vec{y}'' ^2 \vec{y}' \times \vec{y}'' + \det(\vec{y}', \vec{y}'', \vec{y}''') \vec{y}'$	bending-energy flow

Table 1. Low-order Hamiltonians and the corresponding flows in the localized-induction hierarchy. Here $\vec{v} \in \mathbb{R}^3$ is a fixed unit vector. The quantities $\mathcal{A}$ and $\mathcal{V}$ are the area and volume vectors, Φ is the holonomy angle, and $\mathcal{B} = \frac{1}{2} \int |\vec{y}''|^2 ds$ is the bending energy.

Thus $J_Y = M_Y^{-1} \Omega$, and its spectrum is obtained from the generalized eigenvalue problem

$$\Omega X = \lambda M_Y X,$$

or equivalently from the similar Euclidean skew-symmetric matrix

$$M_Y^{-1/2} \Omega M_Y^{-1/2}.$$

Since $M_Y^{-1/2} \Omega M_Y^{-1/2}$ is real and skew-symmetric, the spectrum of J_Y is purely imaginary. Fig. 14 shows it under refinement for three curves⁶. The eigenvalues separate into the three groups predicted by the pointwise model, one near 0 for the tangential reparametrization directions and two approaching $\text{Im } \lambda = \pm i$ for the normal planes.

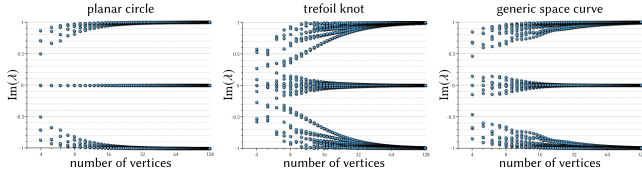


Fig. 14. The spectrum of J_Y is purely imaginary, and for a planar circle, a trefoil knot, and a generic space curve alike it splits under refinement into N eigenvalues near 0 along the discrete reparametrization directions and $2N$ eigenvalues in two branches approaching $\pm i$, one for each normal plane. On the planar circle the near-zero group vanishes to machine precision at every resolution, whereas on the two space curves it only contracts under refinement.

6.1.3 Reparametrization directions. The near-zero cluster has a direct geometric interpretation. In the smooth theory, the tangential fields $X = fT$ are precisely the infinitesimal reparametrizations and form the zero eigenspace of J_Y , while the normal fields give the two branches at $\pm i$.

An N -vertex polygon has no canonical tangent at its vertices. Nevertheless, J_Y has N small eigenvalues, one scalar degree of freedom per vertex. We take the corresponding spectral subspace $\mathcal{R}_Y \subset T_Y \mathcal{M}_g$ as the discrete reparametrization directions.

Planar polygons are an exception. All edges lie in a common plane, so Ω sends every in-plane field to a multiple of the unit normal $\vec{n}$ of said plane. This maps $2N$ dimensions into N and leaves $\ker J_Y = \ker \Omega$ at least N -dimensional. We observe this for the planar circle in Fig. 14, whose near-zero cluster sits at machine precision at every resolution.

⁶The curves are $\gamma(t) = (\cos t, \sin t, 0)$, $((2 + \cos(3t + \frac{3}{10})) \cos 2t, (2 + \cos(3t + \frac{3}{10})) \sin 2t, \sin(3t + \frac{3}{10}))$, and $(\cos t, \sin t, \frac{7}{20} \sin(2t + \frac{7}{10}) + \frac{1}{4} \sin(3t + \frac{3}{10}))$, sampled uniformly in t .

Removing $\mathcal{R}_Y$ is the discrete analogue of passing to shape space. In the Hamiltonian constructions below, $\mathcal{R}_Y$ is removed for the vortex-filament flow, whereas the tangent flow selects a distinguished edge-length-preserving direction associated with this near-kernel. The mKdV flow is obtained from an edge-length-preserving weighted least-squares solve.

6.2 The localized-induction hierarchy

We now derive several Hamiltonian curve flows from the Marsden–Weinstein form. In the smooth setting, these flows belong to the well-studied integrable localized-induction hierarchy [Chern et al. 2020; Langer and Singer 1996] (Table 1), making them natural benchmarks for transferring the Hamiltonian construction to polygonal curves.

For a Hamiltonian H , its vector field X_H is defined by

$$\iota_{X_H} \Omega_Y^{MW} = dH_Y. \quad (25)$$

Because Ω^{MW} is pre-symplectic, X_H is determined only modulo $\ker \Omega_Y^{MW}$, the reparametrization directions. The low-order cases used below are listed in Table 1.

For a constant Hamiltonian, $dH = 0$, so its Hamiltonian vector fields lie in $\ker \Omega_Y^{MW}$. The choice

$$X_0 = \vec{y}'$$

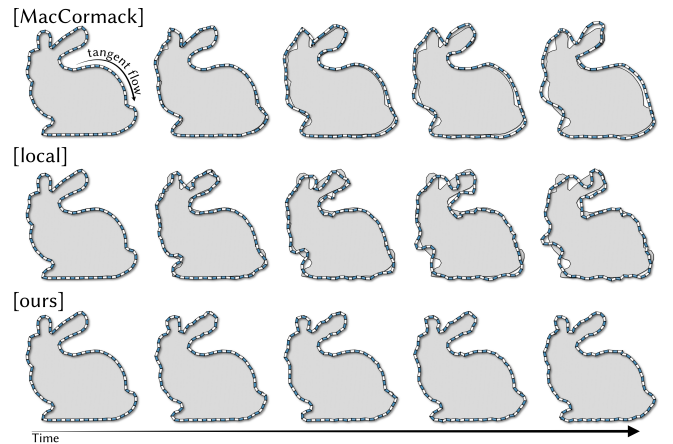


Fig. 15. On the planar bunny, MacCormack advection and the local Möbius flow visibly deform the shape over long times, while the Hamiltonian tangent flow largely preserves its geometric features.

gives the tangent flow, which only reparametrizes the curve and leaves its geometry unchanged (Figs. 15 and 16).

The length functional generates the localized-induction, or vortex-filament, flow (Figs. 17 and 21)

$$X_L = \vec{\gamma}' \times \vec{\gamma}''.$$

The highest-order Hamiltonian in this hierarchy that remains naturally defined on polygons is the holonomy Φ , which generates the modified Korteweg–de Vries flow (Fig. 22)

$$X_\Phi = -\vec{\gamma}''' - \frac{3}{2}|\vec{\gamma}''|^2\vec{\gamma}'.$$

6.3 Hamiltonian flows on discrete curves

Although the smooth tangent, vortex-filament, and mKdV flows involve tangents, curvature, and higher derivatives, their Hamiltonians admit natural polygonal counterparts. For a discrete Hamiltonian H , we therefore define its polygonal vector field by

$$\Omega_Y^{\text{MW}} X_H = dH_Y. \quad (26)$$

The discrete Marsden–Weinstein form does not exactly reproduce the reparametrization kernel of its smooth counterpart. Its near-kernel $\mathcal{R}_Y$ makes Eq. (26) ill-conditioned along its small singular modes. These modes stay separated from the rest of the spectrum under refinement (Fig. 14). For the tangent flow, we select a distinguished direction associated with this near-kernel. For the vortex-filament flow, we remove $\mathcal{R}_Y$, while for the mKdV flow, we impose edge-length preservation in a weighted least-squares solve (Sections 6.3.1–6.3.3). These choices make the corresponding discrete vector fields numerically tractable.

The next member of the hierarchy, generated by the bending energy, lies outside this intrinsic construction because it requires an auxiliary curvature discretization.

6.3.1 Tangent flow. For a constant Hamiltonian, the Hamiltonian equation reduces to

$$\Omega_Y^{\text{MW}} X_0 = 0. \quad (27)$$

Since this homogeneous equation does not canonically select a nonzero discrete field, we impose unit norm and infinitesimal edge-length preservation and minimize its residual. Let C_Y denote the edge-length differential. We define

$$X_0 := \underset{\substack{X \in \ker C_Y \\ \|X\|_{M_Y} = 1}}{\operatorname{argmin}} \|\Omega_Y^{\text{MW}} X\|_{M_Y^{-1}}^2.$$

Here M_Y represents the L^2 metric from Eq. (24), and M_Y^{-1} represents its dual metric on covectors. Thus X_0 is the unit edge-length-preserving field of minimal Marsden–Weinstein residual.

Equivalently, if Q spans $\ker C_Y$ and $X = Qb$, then b is a generalized eigenvector corresponding to the smallest eigenvalue ν :

$$Q^\top \Omega^\top M_Y^{-1} \Omega Q b = \nu Q^\top M_Y Q b.$$

We normalize $X = Qb$ by $\|X\|_{M_Y} = 1$ and fix its sign by positive average alignment with the oriented polygon.

The construction requires no vertex tangents and is global, since X_0 depends on the full matrix Ω , the edge-length constraints, and the chosen metric rather than on a local vertex stencil.

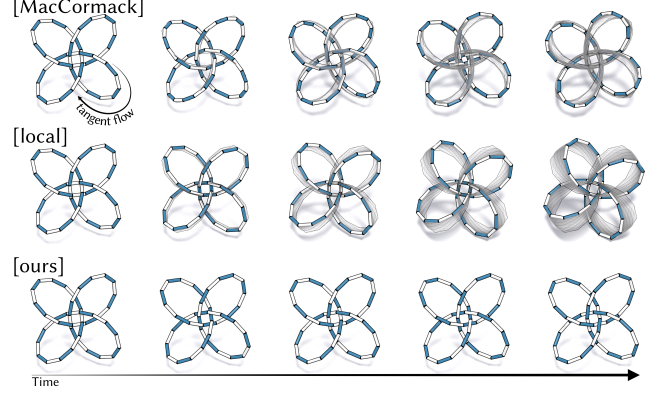


Fig. 16. On a low-resolution fourfold-symmetric elastica, MacCormack advection visibly deforms the curve, while both it and the local Möbius flow accumulate drift. The Hamiltonian tangent flow preserves the shape with substantially less long-time drift.

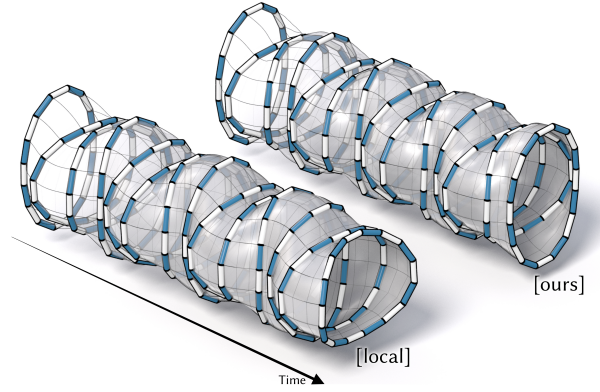


Fig. 17. Evolution of an initially elliptical polygon under the vortex-filament flow. The motion of the Hamiltonian flow exhibits the expected binormal-curvature behavior and remains consistent with the local integrable model.

6.3.2 Vortex-filament flow. For the length Hamiltonian, Eq. (26) becomes

$$\Omega_Y^{\text{MW}} X_L = dL_Y. \quad (28)$$

The near-kernel $\mathcal{R}_Y$ makes direct inversion ill-conditioned, so we solve on its spectral complement.

Let Ω denote the matrix of Ω_Y^{MW} , and let $M_Y = K_Y K_Y^\top$ be a Cholesky factorization. In mass coordinates, write

$$K_Y^{-1} \Omega K_Y^{-\top} = U \Sigma V^\top,$$

with singular values in increasing order. Discarding the first N modes and denoting the retained blocks by $U^\perp$, $\Sigma^\perp$, and $V^\perp$ gives the quotient vortex-filament field

$$X_L^\perp = K_Y^{-\top} V^\perp (\Sigma^\perp)^{-1} (U^\perp)^\top K_Y^{-1} dL_Y.$$

Here $(\cdot)^\perp$ denotes removal of $\mathcal{R}_Y$, not a pointwise normal projection.

This defines a discrete smoke-ring flow on polygonal shape space. Removing $\mathcal{R}_Y$ suppresses reparametrization drift and isolates the shape-changing component up to the near-kernel residual. Like the

tangent flow, the construction is global: the velocity depends on the full mass-coordinate matrix and its singular modes, rather than on a local vertex stencil.

For visualization and time stepping, $X_L^\perp$ may be normalized to unit M_Y -norm. This changes only the time parametrization, not the trajectory in shape space.

6.3.3 mKdV flow. The holonomy Φ is the last Hamiltonian in the hierarchy naturally defined on polygons. Its smooth Hamiltonian field is the geometric mKdV flow, the highest-order flow considered here. Following Bergou et al. [2008], we define polygonal holonomy by discrete parallel transport. Assuming adjacent edges are not antiparallel, a normal vector is transported across each vertex by the minimal rotation mapping the incoming unit edge direction to the outgoing one. Equivalently, this is Levi-Civita parallel transport along the shorter great-circle arc joining the two edge directions on S^2 , making it the canonical discrete counterpart of Bishop transport. Composing these rotations and comparing the transported normal with the terminal reference normal defines $\Phi \in \mathbb{R}/2\pi\mathbb{Z}$. For closed polygons, the endpoint reference normals are identified. For quasi-periodic polygons, the terminal reference is obtained from the initial one by the rotational part of the monodromy. Changing the references shifts Φ only by a constant, so $d\Phi_Y$ is reference-independent [Pinkall and Gross 2024].

We define the discrete mKdV field by the edge-length-preserving weighted least-squares problem

$$X_\Phi := \operatorname{argmin}_{X \in \ker C_Y} \left\| \Omega_Y^{\text{MW}} X - d\Phi_Y \right\|_{M_Y^{-1}}^2.$$

Using a basis of $\ker C_Y$, we eliminate the constraint and solve the reduced weighted least-squares problem directly, without forming normal equations. For integration, we normalize X_Φ to unit M_Y -norm, fixing its speed without changing its direction.

6.3.4 Discussion. In their classical PDE form, the tangent, vortex-filament, and mKdV flows depend explicitly on derivatives of the curve, up to third order in the mKdV case. These derivatives have no canonical meaning on polygons, making the flows difficult to discretize and simulate directly. The Hamiltonian formulation bypasses this obstacle because the relevant Hamiltonians have natural polygonal counterparts. The same discrete Marsden–Weinstein structure that determines the isoperimetric constraints for discrete elastica in Section 4 now converts the Hamiltonian differentials into equations for the corresponding polygonal vector fields.

7 Results and Validation

We evaluate the discrete theory developed in Section 4 from three complementary perspectives. First, we assess its qualitative fidelity by computing discrete elastica across a range of geometric settings and resolutions. Second, we quantitatively test the consistency of the variational formulation and the discrete Marsden–Weinstein structure and benchmark its performance against a baseline formulation. Third, we explore its connections to dynamical systems through Kirchhoff’s analogy and Hamiltonian curve flows, as well as to surface theory.

7.1 Comparison of Optimization Approaches

The isoperimetric formulation computes both closed and quasi-periodic discrete elastica without requiring a curvature discretization, material frame, rod energy, or underlying integrable system. Figures 1, 3, 4, 5, 8, 12, 16, 21, and 23 span a range of resolutions and monodromy types. Since the monodromy enters only through the constraint map, the same augmented Lagrangian solver handles both closed and quasi-periodic curves, with the fundamental-domain vertex positions as optimization variables.

All experiments were implemented in MATLAB [The MathWorks Inc. 2025] and run on a MacBook Pro with an Apple M1 Max processor.

Convergence. We observe convergence under refinement at two levels. First, our curvature-free discrete elastica converge quadratically to the smooth reference solution, as expected for piecewise-linear finite elements (Fig. 13). Second, for polygonal curve theory in general, the eigenvalues of the L^2 -representative J_Y of the discrete Marsden–Weinstein form separate into N eigenvalues near 0 and two normal branches approaching $\pm i$ (Fig. 14).

Performance. The computational advantage of the isoperimetric formulation is already visible in its conditioning (Fig. 13), which compares the reduced Hessians of our formulation and the bending-energy baseline on the same closed elastica under refinement. Here, reduced means the second variation restricted to the constraint tangent space, with the remaining degenerate directions removed. No solver parameters or tolerances enter, so the comparison isolates the two variational formulations.

Fig. 18 reports the runtime under refinement of Alg. 1, together with energy decrease, constraint violation, and first-order optimality over the runtime. We compare our augmented Lagrangian solver for the isoperimetric formulation against two baselines based on the classical bending-energy formulation. For each resolution N , all methods are initialized independently from the same circle with small random perturbations of the vertex positions, use identical solver tolerances,

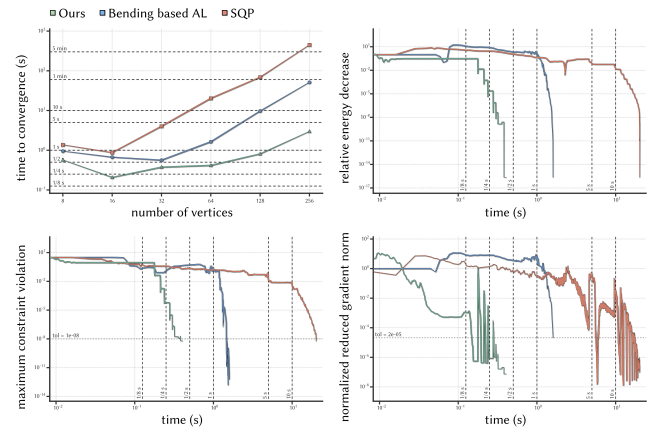


Fig. 18. The isoperimetric formulation is solved with an augmented Lagrangian method and compared to bending-energy baselines. The plots report runtime under refinement, as well as energy decrease, constraint violation, and first-order optimality for an example with $N = 64$ vertices.

and are supplied with analytic gradients for the objective and all constraints. Thus, every resolution is solved from scratch rather than warm-started from a preceding solution. The isoperimetric formulation minimizes L at prescribed $A = V = (0, 0, \pi)$, with convergence histories shown for $N = 64$ and $g = \text{id}$. For the classical formulation, we discretize bending energy following the discrete elastic rods model of Bergou et al. [2008] and impose matching constraints on length and holonomy $\Phi = \pi$, chosen so that the same configuration is stationary. We solve this problem using either an analogous augmented Lagrangian method or the SQP solver provided by Matlab's `fmincon`.

Even with a standard augmented Lagrangian implementation [Nocedal and Wright 2006], the isoperimetric formulation shows favorable convergence and scaling under refinement, consistent with its lower-order variational structure and the absence of curvature estimates. Under this uniform cold-start protocol, all but one of the reported isoperimetric solves finish in under one second, amounting to near-real-time performance for these problem sizes even without warm starts. A faster implementation instead warm-starts each solve from the preceding solution, yielding substantially shorter runtimes in practice. These results are obtained without explicitly removing all gauge freedom, leaving residual indeterminacies that can complicate the numerical solve.

7.2 Connections to dynamical systems

As throughout the paper, elastica are closely connected to several classical dynamical systems. Specifically, we use the Kirchhoff analogy to construct discrete pendulum dynamics from our discrete elastica and illustrate Hamiltonian curve flows generated by the discrete Marsden–Weinstein form.

7.2.1 Kirchhoff analogy. The classical Kirchhoff analogy identifies elastic curves with spinning-top dynamics, whose reduced motion is described by a pendulum equation. Figs. 2 and 19 show that this picture remains visible in our discrete construction. The curves are obtained by solving the discrete isoperimetric problem with prescribed quasi-periodicity. Their edge tangents form a discrete sequence on the unit sphere whose phase portrait follows the corresponding pendulum orbit. The monodromy enforces exact closure or quasi-periodicity of this discrete orbit, so the phase portrait has no drift from time stepping. Thus the pendulum structure is recovered directly from the polygonal elastica solve, without a separate pendulum integrator.

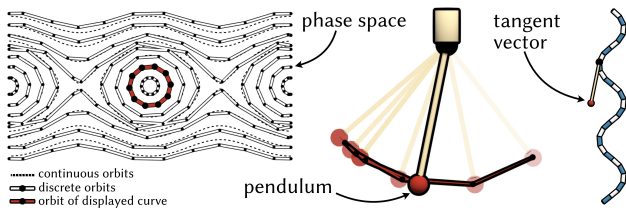


Fig. 19. Solving the discrete isoperimetric problem with prescribed quasi-periodicity recovers the discrete Kirchhoff picture without integrating a pendulum equation. The edge tangents form a discrete orbit in the corresponding pendulum phase portrait.

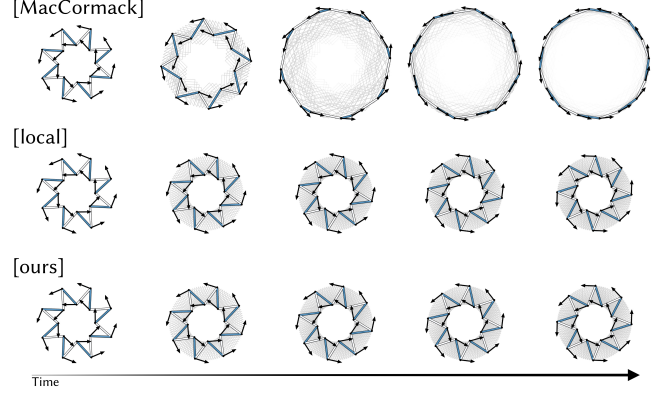


Fig. 20. On a planar star-shaped curve, the Hamiltonian tangent flow agrees closely with the local Möbius flow and preserves the shape, while MacCormack advection visibly deforms the curve over time.

7.2.2 Hamiltonian flows. We compare the Hamiltonian flows induced by the discrete Marsden–Weinstein form to local and sampling-based flow discretizations.

Tangent flow. For tangential motion, a direct approach treats the polygon as sampled data and solves the advection equation

$$\partial_t \gamma - \partial_x \gamma = 0$$

on the vertex grid, for example with semi-Lagrangian, Lax–Friedrichs, Lax–Wendroff, or MacCormack advection. These schemes are simple, but depend on the sampling and do not define an intrinsic polygonal flow.

A geometric alternative comes from the integrable geometry of discrete curves [Doliwa and Santini 1999; Bobenko and Suris 2000]. For an arc-length polygon with unit edges $\vec{e}_i = \vec{\gamma}_{i+1} - \vec{\gamma}_i$, a Möbius-geometric tangent flow is

$$\dot{\gamma}_i = \frac{\vec{e}_{i-1} + \vec{e}_i}{1 + \langle \vec{e}_{i-1}, \vec{e}_i \rangle}.$$

This flow is local, geometric, and edge-length preserving, but may exhibit drift over long time horizons.

Our tangent flow is instead selected spectrally as the edge-length-preserving direction of smallest Marsden–Weinstein residual. Its reparametrization-like character is encoded in the near-kernel structure of the discrete two-form. In the fourfold-symmetric example of Fig. 16, our flow exhibits substantially less drift over long time scales, even at low resolution. For the planar star-shaped curve in Fig. 20, it agrees closely with the local Möbius flow and keeps the shape nearly unchanged, while MacCormack advection visibly smooths the curve over time. On the generic planar bunny in Fig. 15, the Hamiltonian tangent flow again preserves the shape more faithfully over long integrations.

Vortex-filament flow. For the vortex-filament flow, the local integrable discretization is the discrete binormal motion

$$\dot{\gamma}_i = \frac{\vec{e}_i \times \vec{e}_{i-1}}{1 + \langle \vec{e}_i, \vec{e}_{i-1} \rangle},$$

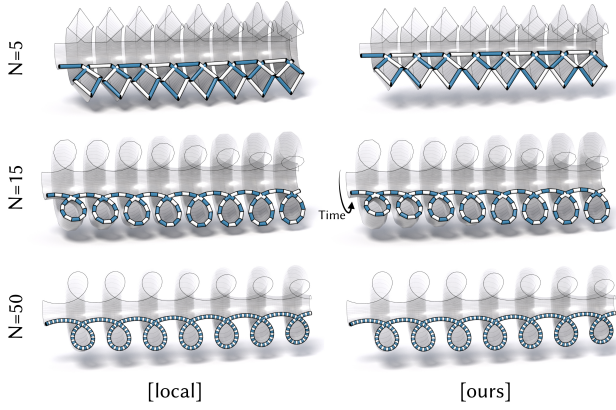


Fig. 21. Since our theory is formulated for quasi-periodic polygonal curves, the Hamiltonian construction applies to both closed curves and open curves with prescribed monodromy. In particular, the evolution of open discrete elastica under the resulting vortex-filament flow is close to a rigid body motion. Since the construction depends only on the N fundamental-domain vertices, the flow can efficiently be computed on a single fundamental piece and extended indefinitely by the monodromy.

again for polygons with unit edge lengths [Doliwa and Santini 1999; Bobenko and Suris 2000; Hoffmann 2008]. This flow is local and integrable, but requires arc-length normalization and becomes singular when adjacent edges are antiparallel. Doubly discrete vortex-filament models based on Darboux or Bäcklund transformations also preserve integrability, but require a choice of transformation parameters together with appropriate closure conditions [Pinkall et al. 2007; Hoffmann 2008].

Our vortex-filament flow is generated by the discrete length Hamiltonian through the discrete Marsden–Weinstein form. For elastic curves, the smooth theory predicts rigid-body motion up to reparametrization. The evolution in Fig. 21 stays close to such a motion and agrees closely, at the shown scale, with the local integrable discretization. For more generic curves, the same construction exhibits the expected binormal-curvature behavior, as shown in Fig. 17.

mKdV flow. Finally, we test two qualitative signatures of the smooth mKdV flow. The flow preserves spherical curves and admits

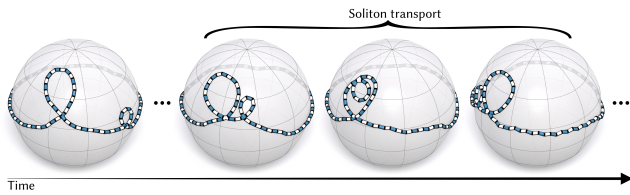


Fig. 22. In the continuous theory, the mKdV flow of a spherical space curve is known to stay on the sphere and to admit soliton solutions, in which a localized shape feature travels along the curve without changing form. Our discrete Hamiltonian counterpart reproduces this behavior over long time horizons without prescribing a separate integrable update.

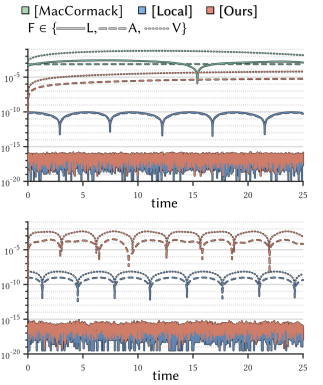
soliton solutions that transport localized geometric features coherently [Langer 1999]. Both features are visible over long time horizons in the simulation shown in Fig. 22. This is particularly striking because the smooth mKdV flow contains the third arclength derivative y''' , which has no classical counterpart on a polygonal curve. Nevertheless, the holonomy angle Φ_Y , the discrete Marsden–Weinstein form, and the edge-length constraint determine a polygonal vector field without requiring a separate discretization of this third-order term.

These experiments highlight the tradeoff. Local formulations are explicit, efficient, and often integrable, but require flow-specific discretization choices. Our construction is global but uniform: the Hamiltonian, the discrete Marsden–Weinstein form, and the relevant constraints determine the vector field.

7.2.3 Numerical consistency. In the smooth hierarchy, the tangent and vortex-filament flows preserve $\mathcal{L}$, $\mathcal{A}$, and $\mathcal{V}$. For the tangent flow, this follows from reparametrization invariance. For the vortex-filament flow, it follows from the Hamiltonian structure and the associated Noether quantities.

Exact preservation need not survive discretization. The local Möbius and binormal flows preserve the corresponding invariants built into their integrable constructions. For our Hamiltonian flows, the preservation defect reflects the failure of the finite-element discretization to reproduce the exact smooth kernel.

The inset shows the normalized instantaneous rates of change of $\mathcal{L}$, $\mathcal{A}$, and $\mathcal{V}$ along the tangent and vortex-filament flows. The local formulations preserve their corresponding invariants to numerical precision, while our flows exhibit small residual rates consistent with the near-kernel mechanism. For the tangent flow, MacCormack advection produces substantially larger rates of change.



7.2.4 Limitations. These experiments also illustrate a limitation of the Hamiltonian-flow formulation. Unlike explicit local models for flows on discrete space curves, the proposed vector fields require a global linear-algebra problem involving the discrete Marsden–Weinstein form, the chosen metric, and, where applicable, the edge-length constraints. This provides a common framework for the different Hamiltonians considered here, but is considerably more expensive than explicit local updates.

The resulting vector fields can also depend on how the near-kernel modes are selected or removed, especially at low resolution or when the small eigenvalues are tightly clustered. In our experiments, this sensitivity decreases under refinement. We therefore view these flows primarily as a structure-driven construction and a validation of the discrete Marsden–Weinstein geometry. They are not intended to replace specialized local integrable schemes when such schemes are available.

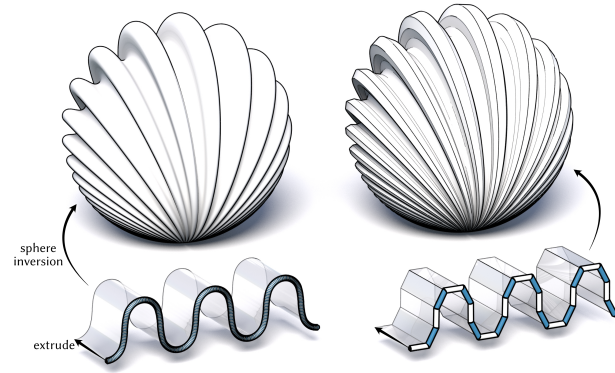


Fig. 23. A cylinder over a planar free elastic curve is a Willmore surface, a property which is preserved under Möbius transformations [Pinkall and Gross 2024]. Applying the same construction to a smooth planar free elastica and to its discrete counterpart gives surfaces with strong resemblance.

7.3 Connections to surface theory

Elastic curves are also connected to the theory of Willmore surfaces [Pinkall 1985; Bohle et al. 2008; Heller 2014; Pinkall and Gross 2024]. A cylinder over a planar free elastic curve is a Willmore surface [Pinkall and Gross 2024]. Since the Willmore property is Möbius invariant, a sphere inversion maps such a cylinder to another Willmore surface.

To construct Fig. 23, we start from a smooth planar free elastica and a discrete counterpart obtained from the isoperimetric optimization. For each generator, we form a ruled cylinder by translating the curve in a fixed normal direction and connecting corresponding samples. We then apply the same sphere inversion pointwise to the smooth and polygonal cylinders. The lower row of Fig. 23 shows the two generators and their extrusions, while the upper row shows the inverted surfaces.

Although this experiment neither defines nor evaluates a discrete Willmore energy, the visual agreement suggests that the curve-level discretization retains sufficient geometric structure to serve as input for analogous (semi-)discrete surface constructions.

8 Discussion & Future Work

We have presented a discrete theory of elastic curves based on the isoperimetric characterization. The definition uses only polygonal length and the monodromy-compatible Noether charges obtained from the area and volume vectors, all of which are well defined directly on polygonal curves. This yields a variational, curvature-free discretization without auxiliary choices such as material frames, rod energies, or underlying integrable systems.

A central advantage of the isoperimetric characterization is its lower differential order. Instead of minimizing a bending energy involving second derivatives and fourth-order stationarity equations, we optimize a first-order functional under low-order constraints. This produces simpler finite-dimensional constrained problems in the vertex positions and avoids estimating curvature from polygonal data. Even at very coarse resolutions, the polygonal solutions exhibit the desired qualitative characteristics and closely resemble their smooth

counterparts. Under refinement they show the expected quadratic convergence for piecewise-linear elements, and the spectrum of the discrete Marsden–Weinstein form splits into N eigenvalues near 0 and two branches approaching $\pm i$.

The construction also retains several classical interpretations of elastica. The Kirchhoff analogy appears through the discrete tangent orbits, and the Marsden–Weinstein form generates Hamiltonian polygonal analogues of the tangent, vortex-filament, and mKdV flows. Möbius images of cylinders over planar free elastica give visually consistent polygonal analogues of Willmore surface constructions. Thus the same discretization keeps the variational, mechanical, Hamiltonian, and surface-theoretic viewpoints connected.

Several questions remain. The splitting of the spectrum of J_y into N eigenvalues near 0 and two branches approaching $\pm i$ awaits a rigorous proof, together with its dependence on element order, sampling, and mass metric. Such a result would in turn control the convergence of the discrete Hamiltonian vector fields built on this spectral gap. The discrete Marsden–Weinstein form may support Hamiltonian constructions beyond the low-order flows considered here. Finally, the surface examples suggest a discrete Willmore theory whose variational meaning remains to be made precise.

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A Normal Form of the Monodromy

Lemma A.1. *For any $g = (R, \vec{b}) \in \text{SE}(3)$ with $R \neq \text{Id}$, there exists $\vec{c} \in \mathbb{R}^3$ such that, in the shifted coordinates $\tilde{\vec{x}} = \vec{x} - \vec{c}$, the monodromy takes the form $\tilde{\vec{x}} \mapsto R\tilde{\vec{x}} + \tilde{\vec{b}}$, where*

$$\begin{cases} R\tilde{\vec{b}} = \tilde{\vec{b}}, & \text{in } 3D, \\ \tilde{\vec{b}} = 0, & \text{in } 2D. \end{cases} \quad (29)$$

PROOF. Setting $\tilde{\vec{b}} = R\vec{c} + \vec{b} - \vec{c}$ and requiring $(R - \text{Id})\tilde{\vec{b}} = 0$ gives $(R - \text{Id})^2\vec{c} = -(R - \text{Id})\vec{b}$. For $R \in \text{SO}(3)$ with $R \neq \text{Id}$, the rotation axis spans $\ker(R - \text{Id})$, so $\text{im}(R - \text{Id})^2 = \text{im}(R - \text{Id})$. Since $-(R - \text{Id})\vec{b} \in \text{im}(R - \text{Id})$, a solution $\vec{c}$ exists, unique up to translation along the axis. In 2D, $R - \text{Id}$ is invertible and $\vec{c} = -(R - \text{Id})^{-1}\vec{b}$ is the unique solution. $\square$

B Closed Forms for Axial Components

When $g \neq \text{id}$ we write $\vec{b} = \ell\vec{s}_g$ with $R\vec{s}_g = \vec{s}_g$, and $\vec{y}^\perp = \vec{y} - \langle \vec{s}_g, \vec{y} \rangle \vec{s}_g$. The components of the completed vectors along the screw axis admit simple closed forms, on which several proofs below rely.

Lemma B.1. *For $\gamma \in \mathcal{M}_g$ with $g \neq \text{id}$, evaluated on any fundamental domain $I = [t_0, \tau(t_0)]$,*

$$\langle \vec{s}_g, \mathcal{A}(\gamma) \rangle = \frac{1}{2} \int_I \langle \vec{s}_g, \vec{y} \times \vec{y}' \rangle ds, \quad \langle \vec{s}_g, \mathcal{V}(\gamma) \rangle = -\frac{1}{2} \int_I |\vec{y}^\perp|^2 \langle \vec{s}_g, \vec{y}' \rangle ds.$$

PROOF. For the area, the endpoint completion in Eq. (8) contributes $-\frac{1}{2} \langle \vec{s}_g, \vec{b} \times \vec{y}(t_0) \rangle = 0$ because $\vec{b} \parallel \vec{s}_g$, which leaves the stated integral.

For the volume, $\vec{y} \times (\vec{y} \times \vec{y}') = \vec{y} \langle \vec{y}, \vec{y}' \rangle - \vec{y}' |\vec{y}|^2$ and $|\vec{y}^\perp|^2 = |\vec{y}|^2 - \langle \vec{s}_g, \vec{y} \rangle^2$ give the pointwise identity

$$\frac{1}{3} \langle \vec{s}_g, \vec{y} \times (\vec{y} \times \vec{y}') \rangle = -\frac{1}{2} |\vec{y}^\perp|^2 \langle \vec{s}_g, \vec{y}' \rangle + \frac{1}{6} \frac{d}{ds} (\langle \vec{s}_g, \vec{y} \rangle |\vec{y}^\perp|^2).$$

Integrating the total derivative over the fundamental domain I , $\frac{1}{6} \langle \vec{s}_g, \vec{y} \rangle |\vec{y}^\perp|^2|_{t_0}^{\tau(t_0)} = \frac{1}{6} \ell |\vec{y}^\perp(t_0)|^2$, where we use $\vec{y}^\perp(\tau(t_0)) = R\vec{y}^\perp(t_0)$ and $\langle \vec{s}_g, \vec{y}(\tau(t_0)) \rangle = \langle \vec{s}_g, \vec{y}(t_0) \rangle + \ell$. The endpoint completion in Eq. (11) cancels it exactly, since $\langle \vec{s}_g, \vec{y}(t_0) \times (\vec{b} \times \vec{y}(t_0)) \rangle = \ell |\vec{y}^\perp(t_0)|^2$, leaving the stated integral. $\square$

C Proof of Lemma 2.3

PROOF. Since τ is a translation, the integral of a τ -periodic density over a fundamental domain $[t_0, \tau(t_0)]$ is independent of t_0 . We use this fact repeatedly, together with $\vec{y}'(\tau(t_0)) = R\vec{y}'(t_0)$.

If $g = \text{id}$, then $\vec{b} = 0$, there is no completion, and $\mathcal{A}(\gamma) = \frac{1}{2} \oint \vec{y} \times \vec{y}'$ has a periodic integrand. Thus the full vector is independent of t_0 .

If $R = \text{Id}$ and $\vec{b} \neq 0$, differentiating the completed vector gives

$$\frac{d}{dt_0} \mathcal{A}(\gamma) = \frac{1}{2} (\vec{\gamma}(\tau(t_0)) - \vec{\gamma}(t_0)) \times \vec{\gamma}'(t_0) - \frac{1}{2} \vec{b} \times \vec{\gamma}'(t_0) = 0,$$

where we use $\vec{\gamma}(\tau(t_0)) - \vec{\gamma}(t_0) = \vec{b}$. Thus the full vector is again independent of t_0 .

If $R \neq \text{Id}$, choose the origin on the screw axis. By Lemma B.1, $\langle \vec{s}_g, \mathcal{A}(\gamma) \rangle = \frac{1}{2} \int_I \langle \vec{s}_g, \vec{\gamma} \times \vec{\gamma}' \rangle ds$. Its integrand is τ -periodic, since $\langle \vec{s}_g, \vec{\gamma}(\tau(t_0)) \times \vec{\gamma}'(\tau(t_0)) \rangle = \langle \vec{s}_g, \vec{\gamma} \times \vec{\gamma}' \rangle$, where we use that $R^\top \vec{s}_g = \vec{s}_g$ and $\vec{b} \parallel \vec{s}_g$. Hence the axial component is independent of t_0 . $\square$

D Proof of Lemma 2.4

PROOF. As in Appendix C, the integral of a τ -periodic density over a fundamental domain is independent of t_0 .

If $g = \text{id}$, then $\vec{b} = 0$, the completion vanishes, and $\mathcal{V}(\gamma) = \frac{1}{3} \oint \vec{\gamma} \times (\vec{\gamma} \times \vec{\gamma}')$ has a periodic integrand. Thus the full vector is independent of t_0 .

If $g \neq \text{id}$, choose $\vec{s}_g$ as in Lemma B.1 (with the origin on the screw axis when $R \neq \text{Id}$). By that lemma $\langle \vec{s}_g, \mathcal{V}(\gamma) \rangle = -\frac{1}{2} \int_I |\vec{\gamma}^\perp|^2 \langle \vec{s}_g, \vec{\gamma}' \rangle ds$. Its integrand is τ -periodic, since $|\vec{\gamma}^\perp(\tau(t_0))|^2 = |R\vec{\gamma}^\perp(t_0)|^2 = |\vec{\gamma}^\perp(t_0)|^2$ and $\langle \vec{s}_g, \vec{\gamma}'(\tau(t_0)) \rangle = \langle \vec{s}_g, R\vec{\gamma}'(t_0) \rangle = \langle \vec{s}_g, \vec{\gamma}'(t_0) \rangle$. Hence the axial component is independent of t_0 . $\square$

E Proof of Theorem 2.5

Lemma E.1. *Let $\gamma \in \mathcal{M}_g$, let $I = [t_0, \tau(t_0)]$ be a fundamental domain, and let $\dot{\gamma}$ be a variation compactly supported in the interior of I . Then, for every $\vec{\lambda}_1, \vec{\lambda}_2 \in \mathbb{R}^3$,*

$$\begin{aligned} d\mathcal{L}(\dot{\gamma}) &= - \int_I \langle \vec{\gamma}'', \dot{\gamma} \rangle ds, \\ d\langle \mathcal{V} | \vec{\lambda}_1 \rangle(\dot{\gamma}) &= \int_I \langle (\vec{\gamma} \times \vec{\lambda}_1) \times \vec{\gamma}', \dot{\gamma} \rangle ds, \\ d\langle \mathcal{A} | \vec{\lambda}_2 \rangle(\dot{\gamma}) &= \int_I \langle \vec{\gamma}' \times \vec{\lambda}_2, \dot{\gamma} \rangle ds. \end{aligned}$$

PROOF. Since $\dot{\gamma}$ is compactly supported in the interior of I , the endpoint corrections in Eq. (8) and Eq. (11) have zero first variation. It therefore suffices to differentiate the integral terms.

The length variation is

$$d\mathcal{L}(\dot{\gamma}) = \int_I \langle \vec{\gamma}', \dot{\gamma}' \rangle ds = - \int_I \langle \vec{\gamma}'', \dot{\gamma} \rangle ds,$$

where the boundary term vanishes because $\dot{\gamma}$ is compactly supported.

For the area component,

$$\begin{aligned} d\langle \mathcal{A} | \vec{\lambda}_2 \rangle(\dot{\gamma}) &= \frac{1}{2} \int_I \langle \vec{\lambda}_2, \dot{\gamma} \times \vec{\gamma}' + \vec{\gamma} \times \dot{\gamma}' \rangle ds \\ &= \frac{1}{2} \int_I \langle \vec{\lambda}_2, \dot{\gamma} \times \vec{\gamma}' - \vec{\gamma}' \times \dot{\gamma} \rangle ds \\ &= \int_I \langle \vec{\gamma}' \times \vec{\lambda}_2, \dot{\gamma} \rangle ds. \end{aligned}$$

For the volume component, writing $\langle \mathcal{V} | \vec{\lambda}_1 \rangle = \frac{1}{3} \int_I \langle \vec{\lambda}_1, \vec{\gamma} \times (\vec{\gamma} \times \vec{\gamma}') \rangle ds$, we compute

$$\begin{aligned} d\langle \mathcal{V} | \vec{\lambda}_1 \rangle(\dot{\gamma}) &= \frac{1}{3} \int_I \langle \vec{\lambda}_1, \dot{\gamma} \times (\vec{\gamma} \times \vec{\gamma}') + \vec{\gamma} \times (\dot{\gamma} \times \vec{\gamma}') + \vec{\gamma} \times (\vec{\gamma} \times \dot{\gamma}') \rangle ds \\ &= \frac{1}{3} \int_I \langle \langle \vec{\gamma}, \vec{\gamma}' \rangle \vec{\lambda}_1 + \langle \vec{\lambda}_1, \vec{\gamma} \rangle \vec{\gamma}' - 2\langle \vec{\lambda}_1, \vec{\gamma}' \rangle \vec{\gamma}, \dot{\gamma} \rangle ds \\ &\quad + \frac{1}{3} \int_I \langle \langle \vec{\lambda}_1, \vec{\gamma} \rangle \vec{\gamma} - |\vec{\gamma}|^2 \vec{\lambda}_1, \dot{\gamma}' \rangle ds. \end{aligned}$$

Integration by parts of the last term and the bac-cab identity then give

$$\begin{aligned} d\langle \mathcal{V} | \vec{\lambda}_1 \rangle(\dot{\gamma}) &= \int_I \langle \langle \vec{\gamma}, \vec{\gamma}' \rangle \vec{\lambda}_1 - \langle \vec{\lambda}_1, \vec{\gamma}' \rangle \vec{\gamma}, \dot{\gamma} \rangle ds \\ &= \int_I \langle (\vec{\gamma} \times \vec{\lambda}_1) \times \vec{\gamma}', \dot{\gamma} \rangle ds, \end{aligned}$$

which proves the claim. $\square$

Lemma E.2. *Let $\gamma \in \mathcal{M}_g$, let $I = [t_0, \tau(t_0)]$, and let $\vec{\lambda}_1, \vec{\lambda}_2$ satisfy the conditions in Theorem 2.5. Then, for every $\dot{\gamma} \in T_\gamma \mathcal{M}_g$,*

$$\begin{aligned} d(\mathcal{L} + \langle \mathcal{V} | \vec{\lambda}_1 \rangle + \langle \mathcal{A} | \vec{\lambda}_2 \rangle)(\dot{\gamma}) \\ = \int_I \langle -\vec{\gamma}'' + (\vec{\gamma} \times \vec{\lambda}_1) \times \vec{\gamma}' + \vec{\gamma}' \times \vec{\lambda}_2, \dot{\gamma} \rangle ds. \end{aligned} \quad (30)$$

PROOF. The interior terms are those computed in Lemma E.1. It remains to show that the boundary terms vanish. Since $\dot{\gamma} \in T_\gamma \mathcal{M}_g$, we have $\dot{\gamma}(\tau(t_0)) = R\dot{\gamma}(t_0)$ and $\vec{\gamma}'(\tau(t_0)) = R\vec{\gamma}'(t_0)$. Thus the boundary term in the length variation is

$$\langle \vec{\gamma}', \dot{\gamma} \rangle|_{t_0}^{\tau(t_0)} = \langle R\vec{\gamma}'(t_0), R\dot{\gamma}(t_0) \rangle - \langle \vec{\gamma}'(t_0), \dot{\gamma}(t_0) \rangle = 0.$$

For the completed area vector, the boundary contribution is

$$\frac{1}{2} \langle \vec{\lambda}_2, \vec{\gamma} \times \dot{\gamma} \rangle|_{t_0}^{\tau(t_0)} - \frac{1}{2} \langle \vec{\lambda}_2, \vec{b} \times \dot{\gamma}(t_0) \rangle, \quad (31)$$

where the second term is the variation of the endpoint correction in Eq. (8). If $R = \text{Id}$, then

$$\langle \vec{\lambda}_2, \vec{\gamma} \times \dot{\gamma} \rangle|_{t_0}^{\tau(t_0)} = \langle \vec{\lambda}_2, \vec{b} \times \dot{\gamma}(t_0) \rangle,$$

so the two terms cancel. If $R \neq \text{Id}$, then $\vec{\lambda}_2$ and $\vec{b}$ are parallel to $\vec{s}_g$, and $R^\top \vec{\lambda}_2 = \vec{\lambda}_2$. Hence Eq. (31) vanishes, since

$$\langle \vec{\lambda}_2, (R\vec{\gamma}(t_0) + \vec{b}) \times R\dot{\gamma}(t_0) \rangle = \langle \vec{\lambda}_2, \vec{\gamma}(t_0) \times \dot{\gamma}(t_0) \rangle,$$

where we use that $\langle \vec{\lambda}_2, \vec{b} \times \dot{\gamma}(t_0) \rangle = \langle \vec{\lambda}_2, \vec{b} \times R\dot{\gamma}(t_0) \rangle = 0$.

For the completed volume vector, the boundary contribution is

$$\begin{aligned} \frac{1}{3} \langle \langle \vec{\lambda}_1, \vec{\gamma} \rangle \vec{\gamma} - |\vec{\gamma}|^2 \vec{\lambda}_1, \dot{\gamma} \rangle|_{t_0}^{\tau(t_0)} \\ - \frac{1}{6} \langle \vec{\lambda}_1, \vec{\gamma}(t_0) \times (\vec{b} \times \vec{\gamma}(t_0)) + \vec{\gamma}(t_0) \times (\vec{b} \times \vec{\gamma}(t_0)) \rangle. \end{aligned}$$

If $g = \text{id}$, then $\vec{b} = 0$, and the endpoint values of both $\vec{\gamma}$ and $\dot{\gamma}$ agree, so this contribution vanishes.

Suppose that $g \neq \text{id}$. Then $\vec{\lambda}_1$ and $\vec{b}$ are parallel to $\vec{s}_g$, while $R^\top \vec{\lambda}_1 = \vec{\lambda}_1$ and $R^\top \vec{b} = \vec{b}$. Substituting the monodromy relations into the first term and expanding gives

$$\langle \langle \vec{\lambda}_1, \vec{\gamma} \rangle \vec{\gamma} - |\vec{\gamma}|^2 \vec{\lambda}_1, \dot{\gamma} \rangle|_{t_0}^{\tau(t_0)} = \langle \vec{\lambda}_1, \vec{b} \rangle \langle \vec{\gamma}(t_0), \dot{\gamma}(t_0) \rangle - \langle \vec{b}, \vec{\gamma}(t_0) \rangle \langle \vec{\lambda}_1, \dot{\gamma}(t_0) \rangle.$$

On the other hand, the bac-cab identity gives

$$\begin{aligned} & \langle \vec{\lambda}_1, \vec{\gamma}(t_0) \times (\vec{b} \times \vec{\gamma}(t_0)) + \vec{\gamma}(t_0) \times (\vec{b} \times \vec{\gamma}(t_0)) \rangle \\ &= 2\langle \vec{\lambda}_1, \vec{b} \rangle \langle \vec{\gamma}(t_0), \vec{\gamma}(t_0) \rangle - 2\langle \vec{b}, \vec{\gamma}(t_0) \rangle \langle \vec{\lambda}_1, \vec{\gamma}(t_0) \rangle. \end{aligned}$$

After multiplication by $1/3$ and $-1/6$, respectively, the two contributions cancel. This proves Eq. (30). $\square$

Lemma E.3. *Let $\gamma \in \mathcal{M}_g$. Then γ is elastic if and only if there exist $\vec{\lambda}_1, \vec{\lambda}_2$ satisfying the conditions in Theorem 2.5 such that*

$$-\vec{\gamma}'' + (\vec{\gamma} \times \vec{\lambda}_1) \times \vec{\gamma}' + \vec{\gamma}' \times \vec{\lambda}_2 = 0. \quad (32)$$

PROOF. Suppose first that Eq. (32) holds. Equivalently, $\vec{\gamma}'' = \vec{\gamma}' \times (\vec{\lambda}_2 - \vec{\gamma} \times \vec{\lambda}_1)$. Setting $\vec{m} = \vec{\lambda}_2 - \vec{\gamma} \times \vec{\lambda}_1$, we find $\vec{\gamma}'' = \vec{\gamma}' \times \vec{m}$ and $\vec{m}' = -\vec{\gamma}' \times \vec{\lambda}_1$. It follows that

$$\frac{d}{ds} \langle \vec{\gamma}', \vec{m} \rangle = \langle \vec{\gamma}' \times \vec{m}, \vec{m} \rangle - \langle \vec{\gamma}', \vec{\gamma}' \times \vec{\lambda}_1 \rangle = 0.$$

Thus $\varrho := \langle \vec{\gamma}', \vec{m} \rangle$ is constant. Crossing $\vec{\gamma}'' = \vec{\gamma}' \times \vec{m}$ with $\vec{\gamma}'$ gives

$$\begin{aligned} \vec{\gamma}' \times \vec{\gamma}'' &= \vec{\gamma}' \times (\vec{\gamma}' \times \vec{m}) = -\vec{m} + \varrho \vec{\gamma}' \\ &= -\vec{\lambda}_2 + \vec{\gamma} \times \vec{\lambda}_1 + \varrho \vec{\gamma}' \\ &= (-\vec{\lambda}_1) \times \vec{\gamma} - \vec{\lambda}_2 + \varrho \vec{\gamma}'. \end{aligned}$$

After renaming the constant vectors, this is Eq. (5), and therefore γ is elastic.

Conversely, let γ be elastic. Then,

$$\vec{\gamma}' \times \vec{\gamma}'' = \vec{c}_1 \times \vec{\gamma} + \vec{c}_2 + \varrho \vec{\gamma}'. \quad (33)$$

If $g = \text{id}$, there are no restrictions on $\vec{c}_1$ or $\vec{c}_2$.

Suppose that $R = \text{Id}$ and $\vec{b} \neq 0$. The left-hand side of Eq. (33), as well as $\vec{\gamma}'$, is periodic. Evaluating the equation at $\tau(t)$ and subtracting its value at t gives $\vec{c}_1 \times \vec{b} = 0$. Thus $\vec{c}_1$ is parallel to $\vec{b}$, while $\vec{c}_2$ is unrestricted.

Finally, suppose that $R \neq \text{Id}$. Evaluating Eq. (33) at $\tau(t)$ and comparing it with R applied to the equation at t gives

$$(\vec{c}_1 - R\vec{c}_1) \times R\vec{\gamma}'(t) + \vec{c}_1 \times \vec{b} + \vec{c}_2 - R\vec{c}_2 = 0. \quad (34)$$

Differentiating in s yields $(\vec{c}_1 - R\vec{c}_1) \times R\vec{\gamma}''(t) = 0$. Unless γ is a straight line, its tangent assumes two nonparallel directions, and hence $\vec{c}_1 = R\vec{c}_1$. The fixed subspace of the nontrivial rotation R is the screw axis, so $\vec{c}_1$ is parallel to $\vec{s}_g$. Since $\vec{b}$ is also parallel to $\vec{s}_g$, Eq. (34) gives $\vec{c}_2 = R\vec{c}_2$, and hence $\vec{c}_2$ is parallel to $\vec{s}_g$. If γ is a straight line, Eq. (32) holds with $\vec{\lambda}_1 = \vec{\lambda}_2 = 0$.

In every case, we may therefore set $\vec{\lambda}_1 = -\vec{c}_1$ and $\vec{\lambda}_2 = -\vec{c}_2$. These vectors satisfy the conditions in Theorem 2.5, and reversing the preceding calculation gives Eq. (32). $\square$

PROOF OF Theorem 2.5. Suppose first that condition (i) holds. Fix a fundamental domain $I = [t_0, \tau(t_0)]$, and let $\dot{\gamma}$ be compactly supported in its interior. Extending it by $\vec{\gamma}(\tau(t)) = R\vec{\gamma}(t)$ gives a smooth variation in $T_\gamma \mathcal{M}_g$. Condition (i) and Lemma E.2 therefore imply

$$\int_I \langle -\vec{\gamma}'' + (\vec{\gamma} \times \vec{\lambda}_1) \times \vec{\gamma}' + \vec{\gamma}' \times \vec{\lambda}_2, \dot{\gamma} \rangle ds = 0.$$

By the fundamental lemma of the calculus of variations, Eq. (32) holds in the interior of I . Since t_0 was arbitrary, it holds on all of $\mathbb{R}$. A partition of unity subordinate to a finite cover of the support by

interiors of fundamental domains, together with Lemma E.1, then proves condition (ii).

Conversely, suppose that condition (ii) holds. Testing against variations compactly supported in the interior of arbitrary fundamental domains and applying Lemma E.1 and the fundamental lemma gives Eq. (32) on all of $\mathbb{R}$. Substitution into Eq. (30) shows that the augmented functional is stationary with respect to every variation in $T_\gamma \mathcal{M}_g$, proving condition (i).

For each fixed pair $(\vec{\lambda}_1, \vec{\lambda}_2)$ satisfying the stated conditions, the preceding argument shows that conditions (i) and (ii) are each equivalent to Eq. (32). By Lemma E.3, γ is elastic if and only if this equation holds for some such pair. Hence γ is elastic if and only if either, and therefore both, of conditions (i) and (ii) holds for some admissible pair. $\square$

F Proof of Theorem 3.1

Throughout, $h = (Q, \vec{c}) \in Z(g)$ and $\gamma \in \mathcal{M}_g$ has monodromy $g = (R, \vec{b})$, evaluated on a fundamental domain $I = [t_0, \tau(t_0)]$. When $g \neq \text{id}$ we write $\vec{b} = \ell \vec{s}_g$ with $R\vec{s}_g = \vec{s}_g$ (so $\ell = |\vec{b}|$ when $R = \text{Id}$) and $\vec{\gamma}^\perp = \vec{\gamma} - \langle \vec{s}_g, \vec{\gamma} \rangle \vec{s}_g$.

By the description of $Z(g)$ in Section 2.3.1, every $h = (Q, \vec{c}) \in Z(g)$ satisfies

$$Q\vec{b} = \vec{b}, \quad R\vec{c} = \vec{c},$$

in each of the three cases, so it suffices to treat the two factors of $h = (\text{Id}, \vec{c}) \circ (Q, 0)$ in turn.

For the rotational factor, $Q \circ \gamma \in \mathcal{M}_g$, and since Q commutes with the cross product and fixes $\vec{b}$, we have $\mathcal{A}(Q \circ \gamma) = Q\mathcal{A}(\gamma)$ and $\mathcal{V}(Q \circ \gamma) = Q\mathcal{V}(\gamma)$.

It therefore suffices to verify the transformation laws for compatible translations, *i.e.*, those with $R\vec{c} = \vec{c}$. We do so for the area (Lemma F.1) and volume (Lemma F.2) separately. The proof of Theorem 3.1 then follows from combining the matching cases.

Lemma F.1. *For $h = (Q, \vec{c}) \in Z(g)$:*

- (i) *if $g = \text{id}$, then $\mathcal{A}(h \circ \gamma) = Q\mathcal{A}(\gamma)$;*
- (ii) *if $R = \text{Id}$, $\vec{b} \neq 0$, then $\mathcal{A}(h \circ \gamma) = Q\mathcal{A}(\gamma) + \vec{c} \times \vec{b}$;*
- (iii) *if $R \neq \text{Id}$, then $\langle \vec{s}_g, \mathcal{A}(h \circ \gamma) \rangle = \langle \vec{s}_g, Q\mathcal{A}(\gamma) \rangle$.*

PROOF. By the reduction it suffices to treat $\gamma \mapsto \gamma + \vec{c}$, $R\vec{c} = \vec{c}$. As $(\gamma + \vec{c})' = \gamma'$ and the monodromy translation is unchanged,

$$\begin{aligned} \mathcal{A}(\gamma + \vec{c}) &= \frac{1}{2} \int_I (\vec{\gamma} + \vec{c}) \times \vec{\gamma}' ds - \frac{1}{2} \vec{b} \times (\vec{\gamma}(t_0) + \vec{c}) \\ &= \mathcal{A}(\gamma) + \frac{1}{2} \vec{c} \times (\vec{\gamma}(\tau(t_0)) - \vec{\gamma}(t_0)) - \frac{1}{2} \vec{b} \times \vec{c}. \end{aligned}$$

With $\vec{\gamma}(\tau(t_0)) - \vec{\gamma}(t_0) = (R - \text{Id})\vec{\gamma}(t_0) + \vec{b}$ and $\frac{1}{2}\vec{c} \times \vec{b} - \frac{1}{2}\vec{b} \times \vec{c} = \vec{c} \times \vec{b}$,

$$\mathcal{A}(\gamma + \vec{c}) = \mathcal{A}(\gamma) + \frac{1}{2} \vec{c} \times (R - \text{Id})\vec{\gamma}(t_0) + \vec{c} \times \vec{b}.$$

If $g = \text{id}$ both added terms vanish. If $R = \text{Id}$ the first vanishes, leaving $\vec{c} \times \vec{b}$. If $R \neq \text{Id}$, pairing with $\vec{s}_g$ annihilates both terms: $\langle \vec{s}_g, \vec{c} \times (R - \text{Id})\vec{\gamma}(t_0) \rangle = \langle \vec{s}_g \times \vec{c}, (R - \text{Id})\vec{\gamma}(t_0) \rangle = 0$ because $\vec{c} \parallel \vec{s}_g$, and $\langle \vec{s}_g, \vec{c} \times \vec{b} \rangle = 0$ because $\vec{b} \parallel \vec{s}_g$. $\square$

Lemma F.2. *For $h = (Q, \vec{c}) \in Z(g)$:*

- (i) *if $g = \text{id}$, then $\mathcal{V}(h \circ \gamma) = Q\mathcal{V}(\gamma) + \vec{c} \times Q\mathcal{A}(\gamma)$;*

- (ii) if $R = \text{Id}$, $\vec{b} \neq 0$, then $\langle \vec{s}_g, \mathcal{V}(h \circ \gamma) \rangle = \langle \vec{s}_g, Q\mathcal{V}(\gamma) + \vec{c} \times Q\mathcal{A}(\gamma) \rangle - \frac{1}{2} |\vec{b}| |\vec{c}^\perp|^2$;
- (iii) if $R = \text{Id}$, $\vec{b} \neq 0$, and $\vec{c}^\perp := \vec{c} - \langle \vec{s}_g, \vec{c} \rangle \vec{s}_g$, then $\langle \vec{s}_g, \mathcal{V}(h \circ \gamma) \rangle = \langle \vec{s}_g, Q\mathcal{V}(\gamma) \rangle$.

PROOF. Again we reduce to the case $\gamma \mapsto \gamma + \vec{c}$, $R\vec{c} = \vec{c}$.

Case (i). Here $\vec{b} = 0$, the completion is absent, and $\oint \vec{\gamma}' = 0$. Expanding $(\vec{\gamma} + \vec{c}) \times ((\vec{\gamma} + \vec{c}) \times \vec{\gamma}')$, the term quadratic in $\vec{c}$ integrates to zero, and the linear term gives, after one integration by parts,

$$\begin{aligned} 3\mathcal{V}(\gamma + \vec{c}) &= 3\mathcal{V}(\gamma) + \oint (\langle \vec{\gamma}, \vec{\gamma}' \rangle \vec{c} - \langle \vec{c}, \vec{\gamma} \rangle \vec{\gamma}') ds + 2\vec{c} \times \mathcal{A}(\gamma) \\ &= 3\mathcal{V}(\gamma) + \oint \frac{1}{2} \langle \vec{\gamma}, \vec{\gamma} \rangle \vec{c} ds - \oint \langle \vec{c}, \vec{\gamma} \rangle \vec{\gamma}' ds + 2\vec{c} \times \mathcal{A}(\gamma) \\ &= 3\mathcal{V}(\gamma) + \frac{1}{2} \oint \vec{c} \times (\vec{\gamma} \times \vec{\gamma}') ds + 2\vec{c} \times \mathcal{A}(\gamma) \\ &= 3\mathcal{V}(\gamma) + \vec{c} \times \mathcal{A}(\gamma) + 2\vec{c} \times \mathcal{A}(\gamma) \\ &= 3\mathcal{V}(\gamma) + 3\vec{c} \times \mathcal{A}(\gamma). \end{aligned}$$

Replacing γ by $Q \circ \gamma$ proves (i).

Cases (ii), (iii). By Lemma B.1, using $\langle \vec{s}_g, (\vec{\gamma} + \vec{c})' \rangle = \langle \vec{s}_g, \vec{\gamma}' \rangle$ and $(\vec{\gamma} + \vec{c})^\perp = \vec{\gamma}^\perp + \vec{c}^\perp$,

$$\begin{aligned} \langle \vec{s}_g, \mathcal{V}(\gamma + \vec{c}) \rangle &= -\frac{1}{2} \int_I |\vec{\gamma}^\perp + \vec{c}^\perp|^2 \langle \vec{s}_g, \vec{\gamma}' \rangle ds \\ &= \langle \vec{s}_g, \mathcal{V}(\gamma) \rangle - \int_I \langle \vec{\gamma}, \vec{c}^\perp \rangle \langle \vec{s}_g, \vec{\gamma}' \rangle ds - \frac{1}{2} |\vec{c}^\perp|^2 \int_I \langle \vec{s}_g, \vec{\gamma}' \rangle ds, \\ &= \langle \vec{s}_g, \mathcal{V}(\gamma) \rangle - \int_I \langle \vec{\gamma}, \vec{c}^\perp \rangle \langle \vec{s}_g, \vec{\gamma}' \rangle ds - \frac{1}{2} \ell |\vec{c}^\perp|^2, \end{aligned}$$

where we used that $\langle \vec{\gamma}^\perp, \vec{c}^\perp \rangle = \langle \vec{\gamma}, \vec{c}^\perp \rangle$ since $\vec{c}^\perp \perp \vec{s}_g$, and $\int_I \langle \vec{s}_g, \vec{\gamma}' \rangle ds = \langle \vec{s}_g, \vec{b} \rangle = \ell$.

Now let $R = \text{Id}$, and consider the integral to be evaluated alongside the one with $\vec{s}_g$ and $\vec{c}^\perp$ exchanged. The product rule for $(\langle \vec{s}_g, \vec{\gamma} \rangle \langle \vec{c}^\perp, \vec{\gamma}' \rangle)'$ gives their sum,

$$\begin{aligned} &\int_I \langle \vec{s}_g, \vec{\gamma} \rangle \langle \vec{c}^\perp, \vec{\gamma}' \rangle ds + \int_I \langle \vec{s}_g, \vec{\gamma}' \rangle \langle \vec{c}^\perp, \vec{\gamma} \rangle ds \\ &= \langle \vec{s}_g, \vec{\gamma} \rangle \langle \vec{c}^\perp, \vec{\gamma} \rangle \Big|_{t_0}^{t(t_0)} = \ell \langle \vec{c}^\perp, \vec{\gamma}(t_0) \rangle, \end{aligned}$$

where we used that $\vec{\gamma}(\tau(t_0)) = \vec{\gamma}(t_0) + \vec{b}$, $\langle \vec{s}_g, \vec{b} \rangle = \ell$, and $\langle \vec{c}^\perp, \vec{b} \rangle = 0$. On the other hand, the Lagrange identity $\langle \vec{s}_g \times \vec{c}^\perp, \vec{\gamma} \times \vec{\gamma}' \rangle = \langle \vec{s}_g, \vec{\gamma} \rangle \langle \vec{c}^\perp, \vec{\gamma}' \rangle - \langle \vec{s}_g, \vec{\gamma}' \rangle \langle \vec{c}^\perp, \vec{\gamma} \rangle$ gives their difference. That is,

$$\begin{aligned} &\int_I \langle \vec{s}_g, \vec{\gamma} \rangle \langle \vec{c}^\perp, \vec{\gamma}' \rangle ds - \int_I \langle \vec{s}_g, \vec{\gamma}' \rangle \langle \vec{c}^\perp, \vec{\gamma} \rangle ds \\ &= 2 \langle \vec{s}_g \times \vec{c}^\perp, \mathcal{A}(\gamma) \rangle + \ell \langle \vec{c}^\perp, \vec{\gamma}(t_0) \rangle. \end{aligned}$$

where we used that $\int_I \vec{\gamma} \times \vec{\gamma}' ds = 2\mathcal{A}(\gamma) + \vec{b} \times \vec{\gamma}(t_0)$. Subtracting the second line from the first cancels the first integral and both endpoint terms, so

$$-\int_I \langle \vec{\gamma}, \vec{c}^\perp \rangle \langle \vec{s}_g, \vec{\gamma}' \rangle ds = \langle \vec{s}_g \times \vec{c}^\perp, \mathcal{A}(\gamma) \rangle = \langle \vec{s}_g, \vec{c} \times \mathcal{A}(\gamma) \rangle,$$

where the last step uses that $\langle \vec{s}_g, \vec{s}_g \times \mathcal{A}(\gamma) \rangle = 0$.

In case (iii) we have $\vec{c} \parallel \vec{s}_g$, so $\vec{c}^\perp = 0$ and $\langle \vec{s}_g, \vec{c} \times \mathcal{A} \rangle = 0$, leaving $\langle \vec{s}_g, \mathcal{V}(h \circ \gamma) \rangle = \langle \vec{s}_g, Q\mathcal{V}(\gamma) \rangle$. $\square$

G Proof of Theorem 3.2

PROOF OF THEOREM 3.2. To show that Ω^{MW} is indeed closed, we make use of the convenient fact that the Marsden–Weinstein form is in fact exact, with the Marsden–Weinstein–Liouville form $\Theta^{MW} \in \Omega^1(\mathcal{M}_g)$ as potential. In what follows we show that

$$\Omega^{MW} = d\Theta^{MW} \in \Omega^2(\mathcal{M}_g).$$

Since the exterior derivative of an exact form always vanishes, this implies Ω^{MW} is closed.

Infinitesimal reparametrizations have the form $\dot{\vec{\gamma}} = f\vec{\gamma}'$, and for such vectors $\det(\dot{\vec{\gamma}}, \vec{\gamma}') \equiv 0$ by alternation of the determinant. Thus they lie in the kernel. Conversely, if $\det(\dot{\vec{\gamma}}, \vec{\gamma}') = \langle \dot{\vec{\gamma}}, \vec{\gamma}' \times \vec{\gamma} \rangle$ vanishes for all $\dot{\vec{\gamma}} \in T_\gamma \mathcal{M}_g$, then $\vec{\gamma}' \times \dot{\vec{\gamma}} = 0$. Hence $\dot{\vec{\gamma}}$ is pointwise parallel to $\vec{\gamma}'$, so $\dot{\vec{\gamma}} = f\vec{\gamma}'$ for some τ -periodic function $f(\tau(t)) = f(t)$. Therefore the kernel consists exactly of the infinitesimal reparametrizations. $\square$

The Marsden–Weinstein–Liouville form is most conveniently constructed by first choosing a primitive of the Euclidean volume form, i.e., some $\beta \in \Omega^2(\mathbb{R}^3)$ such that $d\beta = \text{vol}$. To be well-defined on $\mathcal{M}_g$, the choice must be compatible with the monodromy of $\mathcal{M}_g$, which is why we additionally require that $g^*\beta = \beta$ (Lemma G.1). Given such a β , we define the Marsden–Weinstein–Liouville form $\Theta^{MW} \in \Omega^1(\mathcal{M}_g)$ by

$$\Theta_Y^{MW}(\dot{\vec{\gamma}}) = \int_Y \iota_{\dot{\vec{\gamma}}} \beta = \int_I \beta(\dot{\vec{\gamma}}, d\vec{\gamma}).$$

Now a direct computation verifies that indeed $d\Theta^{MW} = \Omega^{MW}$. Let $\dot{\vec{\gamma}}$ and $\dot{\vec{\gamma}}$ be local commuting variation fields on $\mathcal{M}_g$. At γ , their pointwise representatives are the sections $\dot{\vec{\gamma}}, \dot{\vec{\gamma}} \in \Gamma(\gamma^* T\mathbb{R}^3)$ in the tangent bundle $T\mathbb{R}^3$ pulled back by γ . With ∇ denoting the flat connection on $\mathbb{R}^3$, applied pointwise to these pullback sections, the bracket term in $d\Theta^{MW}$ vanishes. Then,

$$\begin{aligned} d\Theta_Y^{MW}(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}) &= d(\Theta^{MW}(\dot{\vec{\gamma}}))_\gamma(\dot{\vec{\gamma}}) - d(\Theta^{MW}(\dot{\vec{\gamma}}))_\gamma(\dot{\vec{\gamma}}) \\ &= \int_I \left[(\nabla_{\dot{\vec{\gamma}}} \beta)(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}) + \beta(\nabla_{\dot{\vec{\gamma}}} \dot{\vec{\gamma}}, \dot{\vec{\gamma}}) + \beta(\dot{\vec{\gamma}}, \nabla_{\dot{\vec{\gamma}}} \dot{\vec{\gamma}}) \right. \\ &\quad \left. - (\nabla_{\dot{\vec{\gamma}}} \beta)(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}) - \beta(\nabla_{\dot{\vec{\gamma}}} \dot{\vec{\gamma}}, \dot{\vec{\gamma}}) - \beta(\dot{\vec{\gamma}}, \nabla_{\dot{\vec{\gamma}}} \dot{\vec{\gamma}}) \right] dt. \end{aligned}$$

Since $\dot{\vec{\gamma}}$ and $\dot{\vec{\gamma}}$ commute, $\nabla_{\dot{\vec{\gamma}}} \dot{\vec{\gamma}} = \nabla_{\dot{\vec{\gamma}}} \dot{\vec{\gamma}}$, so the middle terms cancel. Integrating the remaining t -derivative terms by parts gives

$$\int_I [\beta(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}) - \beta(\dot{\vec{\gamma}}, \dot{\vec{\gamma}})] dt = \int_I (\nabla_{\dot{\vec{\gamma}}} \beta)(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}) dt,$$

with no boundary term because the endpoints are related by g and $g^*\beta = \beta$. Hence, using that $d\beta = \text{vol} = \det$, we find

$$\begin{aligned} d\Theta_Y^{MW}(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}) &= \int_I \left[(\nabla_{\dot{\vec{\gamma}}} \beta)(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}) - (\nabla_{\dot{\vec{\gamma}}} \beta)(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}) + (\nabla_{\dot{\vec{\gamma}}} \beta)(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}) \right] dt \\ &= \int_I d\beta(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}, \dot{\vec{\gamma}}) dt = \int_I \det(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}, \dot{\vec{\gamma}}) dt = \Omega_Y^{MW}(\dot{\vec{\gamma}}, \dot{\vec{\gamma}}). \end{aligned}$$

Lemma G.1. Θ^{MW} is well-defined on $\mathcal{M}_g$ if and only if $g^*\beta = \beta$.

PROOF. Let $\gamma \in \mathcal{M}_g$ and $\dot{\vec{\gamma}} \in T_\gamma \mathcal{M}_g$. Well-definedness requires that $\Theta_Y^{MW}(\dot{\vec{\gamma}})$ not depend on which fundamental domain one integrates over. For $t \in \mathbb{R}$, write the potential evaluated on $[t, \tau(t)]$

as

$$\Theta^{[t]} := \int_t^{\tau(t)} \beta_{\tilde{\gamma}(s)}(\dot{\tilde{\gamma}}(s), \tilde{\gamma}'(s)) ds,$$

so that independence amounts to $\Theta^{[t_0]} = \Theta^{[t_1]}$ for all $t_0, t_1 \in \mathbb{R}$. Since τ is a translation, $\tau' \equiv 1$, and the Leibniz rule gives

$$\frac{d}{dt} \Theta^{[t]} = \beta_{\tilde{\gamma}(\tau(t))}(\dot{\tilde{\gamma}}(\tau(t)), \tilde{\gamma}'(\tau(t))) - \beta_{\tilde{\gamma}(t)}(\dot{\tilde{\gamma}}(t), \tilde{\gamma}'(t)).$$

The quasi-periodicity $\tilde{\gamma}(\tau(t)) = g(\tilde{\gamma}(t))$ gives, upon differentiation in t , $\tilde{\gamma}'(\tau(t)) = dg \tilde{\gamma}'(t)$ and $\dot{\tilde{\gamma}}(\tau(t)) = dg \dot{\tilde{\gamma}}(t)$. Hence the first term equals

$$\beta_{g(\tilde{\gamma}(t))}(dg \dot{\tilde{\gamma}}(t), dg \tilde{\gamma}'(t)) = (g^* \beta)_{\tilde{\gamma}(t)}(\dot{\tilde{\gamma}}(t), \tilde{\gamma}'(t)),$$

so that

$$\frac{d}{dt} \Theta^{[t]} = (g^* \beta - \beta)_{\tilde{\gamma}(t)}(\dot{\tilde{\gamma}}(t), \tilde{\gamma}'(t)).$$

This vanishes for all $\gamma \in \mathcal{M}_g$, all $\dot{\gamma} \in T_\gamma \mathcal{M}_g$, and all t if and only if $g^* \beta = \beta$. Therefore $\Theta^{[t_0]} = \Theta^{[t_1]}$ for all t_0, t_1 . That is, Θ^{MW} is independent of the chosen fundamental domain if and only if $g^* \beta = \beta$.

The Marsden–Weinstein–Liouville form Θ^{MW} may depend on the chosen admissible primitive β . Its exterior derivative does not: for every admissible β , the calculation above gives

$$d\Theta^{MW} = \Omega^{MW}.$$

Consequently, the Marsden–Weinstein–Liouville forms associated with any two admissible primitives differ by a closed one-form on $\mathcal{M}_g$, and the resulting Marsden–Weinstein form is independent of this choice. $\square$

Example G.2. An admissible primitive can be chosen explicitly. For $g = \text{id}$, let

$$\beta_0 = \frac{1}{3} \star \vec{p}^b, \quad \vec{p}(\vec{x}) = \vec{x}.$$

For translational or screw monodromy with axis $\mathbb{R}\vec{s}_g$, define

$$\vec{p}_{\vec{s}_g}(\vec{x}) := \vec{x} - \langle \vec{x}, \vec{s}_g \rangle \vec{s}_g$$

and set

$$\beta_{\vec{s}_g} = \frac{1}{2} \star \vec{p}_{\vec{s}_g}^b.$$

Then $d\beta_{\vec{s}_g} = \text{vol}$ and $g^* \beta_{\vec{s}_g} = \beta_{\vec{s}_g}$. In either case, $d\beta = \text{vol}$ and $g^* \beta = \beta$.

H Proof of Lemma 3.3

PROOF. Let $h = (Q, \vec{c}) \in Z(g)$. Since h commutes with g ,

$$h(\gamma)(\tau(t)) = h(g(\gamma(t))) = g(h(\gamma(t))),$$

so h maps $\mathcal{M}_g$ to itself. Its differential is $dh(\dot{\gamma}) = Q\dot{\gamma}$. Hence, using Eq. (13),

$$\begin{aligned} (h^* \Omega^{MW})_\gamma(\dot{\gamma}, \dot{\gamma}) &= \Omega_{h(\gamma)}^{MW}(Q\dot{\gamma}, Q\dot{\gamma}) \\ &= \int_I \det(Q\dot{\gamma}, Q\dot{\gamma}, Qd\vec{\gamma}) = \Omega_Y^{MW}(\dot{\gamma}, \dot{\gamma}), \end{aligned}$$

since $Q \in \text{SO}(3)$. Thus $Z(g)$ preserves Ω^{MW} . Moreover, rigid motions commute with reparametrizations, so the action descends to the quotient. Since Ω^{MW} descends there to a symplectic form by Theorem 3.2, the induced $Z(g)$ -action is symplectic. $\square$

I Proof of Theorem 3.4

PROOF. Let $(\vec{\omega}, \vec{v}) \in \mathfrak{z}(g)$. Its infinitesimal vector field is

$$X_{(\vec{\omega}, \vec{v})}(\gamma) = \vec{\omega} \times \vec{\gamma} + \vec{v}.$$

By the definition of the Marsden–Weinstein form,

$$\iota_{X_{(\vec{\omega}, \vec{v})}} \Omega_Y^{MW}(\dot{\gamma}) = \int_I \det(\vec{\omega} \times \vec{\gamma}, \dot{\gamma}, \tilde{\gamma}') ds + \int_I \det(\vec{v}, \dot{\gamma}, \tilde{\gamma}') ds.$$

Differentiating $\mathcal{A}$ or $\mathcal{V}$ against a general variation $\dot{\gamma} \in T_\gamma \mathcal{M}_g$ produces boundary terms at the seam at t_0 and $\tau(t_0)$. These are canceled by the first variations of the endpoint completions in Eq. (8) and Eq. (11), as shown in the proof of Lemma E.2 (whose compatibility conditions on $(\vec{\lambda}_1, \vec{\lambda}_2)$ are precisely those defining $(\vec{\omega}, \vec{v}) \in \mathfrak{z}(g)$). Hence, for every $\dot{\gamma} \in T_\gamma \mathcal{M}_g$,

$$\begin{aligned} \int_I \det(\vec{v}, \dot{\gamma}, \tilde{\gamma}') ds &= \int_I \langle \tilde{\gamma}' \times \vec{v}, \dot{\gamma} \rangle ds = d\langle \vec{v}, \mathcal{A} \rangle(\dot{\gamma}), \\ \int_I \det(\vec{\omega} \times \vec{\gamma}, \dot{\gamma}, \tilde{\gamma}') ds &= \int_I \langle (\tilde{\gamma} \times \vec{\omega}) \times \tilde{\gamma}', \dot{\gamma} \rangle ds = d\langle \vec{\omega}, \mathcal{V} \rangle(\dot{\gamma}). \end{aligned}$$

Hence

$$\iota_{X_{(\vec{\omega}, \vec{v})}} \Omega_Y^{MW}(\dot{\gamma}) = d\langle (\mathcal{V}(\gamma), \mathcal{A}(\gamma)) | (\vec{\omega}, \vec{v}) \rangle(\dot{\gamma}),$$

and restriction from $\mathfrak{se}(3)$ to $\mathfrak{z}(g)$ yields the claim. $\square$

J Proof of Theorem 4.3

PROOF. Let $\mathcal{I}$ be the piecewise-linear interpolation map. By construction,

$$\Omega^{MW} = \mathcal{I}^* \Omega^{MW}, \quad \mathcal{A} = \mathcal{I}^* \mathcal{A}, \quad \mathcal{V} = \mathcal{I}^* \mathcal{V}.$$

Let $\xi = (\vec{\omega}, \vec{v}) \in \mathfrak{z}(g)$. The smooth infinitesimal rigid-motion field is $X_\xi(\gamma) = \vec{\omega} \times \vec{\gamma} + \vec{v}$. Its discrete counterpart $X_\xi^{\text{disc}}(\gamma)$ is obtained by applying the same formula at the vertices. Since this formula is affine in $\vec{\gamma}$, interpolation of the vertexwise field agrees with the smooth field along the interpolated curve:

$$d\mathcal{I}_\gamma(X_\xi^{\text{disc}}(\gamma)) = X_\xi(\mathcal{I}(\gamma)).$$

Consequently, using Theorem 3.4,

$$\iota_{X_\xi^{\text{disc}}} \Omega^{MW} = \mathcal{I}^* \left(\iota_{X_\xi} \Omega^{MW} \right) = \mathcal{I}^* d\langle (\mathcal{V}, \mathcal{A}) | \xi \rangle = d\langle (\mathcal{V}, \mathcal{A}) | \xi \rangle,$$

which yields the claim. $\square$